NUMERICAL INVESTIGATION OF OPTION PRICING USING BLACK-SCHOLES-MERTON PARTIAL DIFFERENTIAL EQUATION WITH TRANSACTION COST AND NON-CONSTANT VOLATILITY

CALVINCE OCHIENG FWAGA

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DECLARATION AND APPROVAL

Declaration

I declare that this thesis is my original work and has not been presented for the conferment of degree or award of diploma in this or any other University whatsoever.

Signature..................................... Date...........................

Calvince Ochieng Fwaga

PGC/AM/001/19

Approval

This thesis has been submitted with our approval as the University Supervisors:

Signature..................................... Date.....................................

Wilys O. Mukuna, PhD

Department of Mathematics, Acturial and Physical Sciences

University of Kabianga

Signature..................................... Date.....................................

Levi O. Otanga, PhD

Department of Mathematics, Acturial and Physical Sciences

University of Kabianga

ABSTRACT

Over the years studies have been done on option pricing valuation. The world market economies have experienced tremendous asset price fluctuations since 1980s. For this reason, efforts have been directed towards developing reliable and more accurate option pricing models due to volatility and unpredictable market forces. Black-Scholes-Merton model has so far been proved to be robust and significant tool for the valuation of an option. To achieve more reliable and accurate price estimates, this study investigated the effects of transaction cost and non-constant volatility on call and put option of an asset price using a two-dimensional Black-Scholes-Merton Partial Differential Equation. The Dufort-Frankel Finite Difference Method was then used to approximate the solution to the Black-Scholes-Merton model equation describing the value of an option with boundary conditions. The simulation was done with the aid of MATLAB software program. The effects of incorporating transaction cost and non-constant volatility on the two assets prices on the value of an option using Black-Scholes-Merton Partial Differential Equation were determined. It was established that as the volatility increases, the call and put option values also increase. Further, the study established that as transaction cost increases, the call and put option values decrease. The effects of incorporating transaction cost and non-constant volatility on the values of call and put option were shown in tabular form and presented graphically. These results will be useful to the investors in computing possible returns on investment based on more accurate asset pricing and to the government on policy formulation in controlling prices in stock exchange market.

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DEDICATION

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I dedicate this thesis to my dear wife Eve Odhiambo, my son Christoffels Ochieng and my daughter Evra Christen.

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LIST OF ABBREVIATIONS

BS	Black-Scholes
BSE	Black-Scholes Equation
BSM	Black-Scholes Model
BSMPDE	Black-Scholes-Merton Partial Differential Equation
FDM	Finite Difference Method
MATLAB	Matrix Laboratory
PDE	Partial Differential Equation
SDE	Stochastic Differential Equation
2D BSMPDE	Two dimensional Black-Scholes-Merton Partial Differential Equation

LIST OF SYMBOLS/NOTATIONS

C_1	Transaction cost of first asset
C_2	Transaction cost of second asset
K	Strike price
r	Risk free interest rate
S	Underlying asset price
S_T	Current asset price at expiration
S_1	Price of first asset
S_2	Price of second asset
T	The expiration date of an option
t	Excise time
V	Value of an option
μ	Drift coefficient
π	Portfolio
ρ	Correlation between the two assets
σ	Volatility of the two assets
σ_1	First asset price volatility
σ_2	Second asset price volatility
$\sum$	Summation
$\frac{\partial V}{\partial t}$	Partial derivative of option value with respect to time

OPERATIONAL DEFINITION OF TERMS

Expiration date

It is the date or time on which an option's right expires and become worth-less if not exercised. There are two basic types of options in relation to expiration date namely; European option and American option.

European option is an option which can only be exercised at the expiry date. It provides the holder a right to sell or buy for example a stock at a defined future time T for a certain price V. On the other hand, American option is an option that can be exercised before or on the expiry date, that is, it can be exercised at any given time t up to and including the expiration time T

Geometric Brownian motion

This is a continuous stochastic process in which the logarithm of the randomly varying quantity follows a Brownian motion.

Option

This is a financial contract that gives its holder the right but not the obligation to buy or sell a certain amount of financial asset by a certain date for a certain future price called the strike price. Options are categorized into two namely; call option and put option.

Call option is a contract which grants its holder the right to buy a certain amount of underlying asset at a strike price at some specified time in the future.

For a call option, if S_T is higher than the strike price K, then the holder of the option can exercise rights to buy the asset at the future price K in order to gain profit.

On the other hand, Put option is a contract which gives its holder the right to sell a certain amount of underlying asset at a strike price at some time in the future. For a put option, if K is greater than S_T then the seller would wish to sell at future higher price K in order to gain profit.

Portfolio

This is a collection of financial assets such as stocks, bonds and cash equivalents held by an investment institution or company.

Risk-less interest rate

This is the annual interest rate of bonds or other risk- free investment.

Strike price

This is the predetermined price of an underlying asset.

Volatility

It is a measure of variation of price of a financial instrument over time. There are two key types of volatility namely; historic volatility and implied volatility.

Historic volatility is a measure of price variation derived from time series of past market prices.

On the other hand, Implied volatility is a measure of variation of price of a financial instrument derived from market price of market traded derivative.

CHAPTER ONE

INTRODUCTION

1.1 Overview

In this chapter, the background of the study is provided. The statement of the problem, general objective, specific objectives, significance of the study and assumptions used are also given.

1.2 Background of the study

Black-Scholes equation is a partial differential equation which describes the price evolution of a European call option, a financial contract that gives the holder the right but not the obligation to buy an underlying asset at a strike price on or before a specified expiration date.

Anwar and Andallah (2018) state that Fischer Black and Myron Scholes initially presented the Black-Scholes model in their groundbreaking 1973 paper, "The Pricing of Options and Cooperate Liabilities," which was published in the Journal of Political Economy. In the same year, they derived a partial differential equation known as the Black-Scholes equation which estimates the price of an option over time. Robert C Merton (1973) later published a paper escalating the Mathematical understanding of the option pricing model created by the term Black-Scholes option pricing model. The equation formed the basis for a revolutionary approach to options pricing that became known as the Black-Scholes-Merton model.

Black-Scholes equation takes into account several factors that affect the value of a call option such as price of the underlying asset, the strike price, the time to expiration, the risk free interest rate and the volatility of the underlying asset. By solving the equation, one can obtain the theoretical value of European call option which can then be compared to market price to determine if it is overpriced or under priced.
Before the discovery of Black-Scholes model by Fischer Black, Myron Scholes and Robert Merton in 1973, there was no standard method of option pricing agreed upon by the option traders. Traders majorly relied on their intuition to price options. The pricing of financial derivatives such as options was relatively unsophisticated and inexact process. Options were typically priced based on intuition and rule of the thumb rather than rigorous mathematical framework.

Moore and Juh (2006) studied, Derivative pricing 60 years before Black-Scholes: Evidence from Johannesburg Stock Exchange. The study employed daily data for warrants traded on the Johannesburg Stock exchange between 1909 and 1922 and for a broker's call option quotes on stocks for 1908 to 1911. They used the data set to test how close derivative prices are to Black-Scholes (1973) prices and to compute profits for investors using a simple trading rule for call options. The study showed that long before the development of the formal theory, investors had an intuitive grasp of the determinants of derivative pricing.

A breakthrough was later seen when Black-Scholes-Merton model became the most robust and significant tool for the valuation of an option. Fluctuation in market prices of assets prompted rigorous mathematical and probabilistic concepts through the theory of stochastic process, also referred to as Wiener process (Ndede et. al. 2019).

The concepts solved the challenges in option valuation and gave new mathematical ideas that provided solutions to problems in finance and other fields. Further studies on stochastic process and stochastic differential equations resulted to laws that govern stochastic integration and solutions to stochastic differential equations (SDEs).

Stochastic differential equations play a key role in understanding random phenomena in various fields such as finance, economics and engineering among others. Particularly in finance, SDEs are used to model for asset price fluctuation. It is an integral equation in Black-Scholes-Merton partial differential equation.

Despite the great role played by the Black-Scholes-Merton model in option valuation, it has a number of weaknesses due to some of its unrealistic assumptions as discussed by Jankova (2018). The study investigated the drawbacks of the Black-Scholes model for option pricing.In this study, the assumptions of the Black-Scholes model were outlined. These assumptions include; the constant volatility, the model assumes that the volatility of the underlying asset is constant over the life of the option. The model also assumes that there is no transaction cost such as taxes and other fees associated with trading the underlying asset or option. The model also assumes that there is continuous trading of the underlying asset without any breaks or interruptions. Again, there is an assumption that the underlying asset does not pay any dividends during the life of the option. The model also assumes that there is no risk free arbitrage opportunity to mean that it is not possible to make a guaranteed profit by buying and selling the underlying asset. Also, there is an assumption that the returns of the underlying asset follow a lognormal distribution to mean that small changes in the price of the underlying asset are more likely than large changes. Additionally, the model assumes that the market is efficient and that the available information is already reflected in the price of the underlying asset.

The study by Jankova (2018) found that the greatest weakness of the model is its assumption of constant volatility which does not practically apply in financial markets. The unrealistic assumption of constant volatility and transaction cost form the basis of this research.

The Black-Scholes model has been very well-liked since its 1973 proposal, as stated by He and Lin (2021). This can be attributed to the model's tractability and simplicity. However, the simplified assumption that the underlying asset price follows a lognormal distribution under this model is inconsistent with the real market observation making it unable to capture the main features such as skewness and fat tails as exhibited by asset returns. Such inconsistencies between the model and the reality has definitely caused pricing biases. This has consequently attracted a lot of research interest into developing various modifications to the Black-Scholes model. One of the most proposed approaches is to introduce an additional stochastic source in modelling the underlying asset by making volatility another random variable.

Following the well-known Black-Scholes formula for pricing call options under constant volatility, Fouque et. al. (2003) indicate that the 1980s and 1990s saw a number of publications prompted by the need for more generic non-constant volatility models in Financial Mathematics. In particular a lot of attention has been paid to stochastic volatility in which volatility is fluctuating randomly driven by an additional Brownian motion. Volatility of an option is determined by a number of factors as discussed by Handayani et. al. (2018). In the research paper, the determinants of stock price volatility in the Indonesian manufacturing sector. Such determining factors include; dividend yield, higher dividend yields tend to decrease volatility as investors are more likely to hold the underlying asset for its income generating potential rather than speculative purposes. Another factor is the interest rate.

Higher interest rate tend to increase volatility as investors become more risk averse and look for higher returns. Again, options with strike price close to the current market price of the underlying asset tend to have higher volatility. Time to expiration is another factor which affects volatility. The longer the time until expiration the more the time the underlying asset has to move in price thereby increasing the potential for volatility.
As the option approaches expiration the potential for volatility decreases.
Additionally, market conditions such as economic events, political changes or natural disasters can create volatility in the underlying asset of an option. Such uncertainties and unexpected events tend to increase volatility. It therefore implies that volatility cannot be constant.

Stochastic volatility is the main concept in the field of financial economics and Mathematical finance to deal with the endemic time varying and co-dependence found in the financial market. A lot has to be done to predict variances.
Asset pricing theory is dominated by the idea that higher rewards may be expected when risks are high but these risks change with time in complicated ways. (Shephard and Andersen 2009). Some of the changes in the level of risk can be modelled stochastically where the level of volatility is allowed to change over time.

A financial mathematical analysis model by Mondal et. al.(2017) to obtain market option prices with changing volatilities revealed that the approximations by the Heston model improve the accuracy of option pricing but experiences challenge with pricing options, which takes longer time to mature. Market stability remains a key factor in financial markets, and therefore concentrating on the volatility of stock is necessary.

The impact of volatility on the expected returns has forced the researchers to give attention to most accurate computation of price of an asset. The Black-Scholes model's pricing accuracy, according to Wattanatorn and Sombultawe (2021), is called into question primarily due to its assumption of constant volatility and zero transaction cost.

A study by Melino and Turnbull (1990) investigated the effects of stochastic volatility in foreign currency option pricing. The study proposed and examined a diffusion model with stochastic volatility. The study revealed that allowing volatility to be stochastic improves the accuracy of the option price prediction. This study forms a basis of our research on option pricing where the volatility is non-constant.

The main focus of financial economics research is on the expected market returns. Scholars have been conducting research to address the instability of the predicted market returns. The stability of market returns is important in investment and this calls for the research on methods that can guarantee more accurate computation of option price with stochastic volatility.

1.3 Statement of the Problem

Inaccuracy in option pricing computation has posed a great challenge in financial markets. The inaccuracy is majorly attributed to the assumption of constant volatility and zero transaction costs as demonstrated in Black-Scholes-Merton model.

Inaccurate valuation of asset price due to constant volatility and zero transaction costs exposes investors to possible financial losses.

Accurate determination of asset price is therefore important for investors in the financial market in achieving the main goal of business which is maximizing on expected return but minimizing risks.

The inaccuracy in option valuation due to constant volatility and zero transaction costs has attracted researchers to develop more accurate option pricing models that focus on estimating asset price. Most of the existing studies have not taken into account the effects of incorporating transaction costs on option pricing. Based on the unrealistic assumptions of the constant volatility and zero transaction costs, this study employs non-constant volatility and transaction costs with a non-linear two-dimensional BSMPDE for a more accurate way of estimating assets prices.

1.4 General Objective

The general objective of the study was to numerically investigate the option pricing using Black-Scholes-Merton Partial Differential Equation with non-constant volatility and transaction cost.

1.5 Specific Objectives

The specific objectives of this study were:

(i) To develop a mathematical model of Black-Scholes-Merton partial differential equation using Dufort-Frankel numerical scheme.

(ii) To determine the effect of varying volatility on call and put option values of asset price.

(iii) To determine the effect of incorporating transaction costs on call and put option values of asset price.

1.6 Justification of the Study

The Black-Scholes-Merton model is based on the assumptions of constant volatility and zero transaction costs among others. The assumption of constant volatility might not hold in all situations. By letting volatility to vary randomly, the model can better capture the market dynamics such as changes in volatility over time.

This can lead to more accurate pricing of options. Additionally, real world trading in most cases involve fees, commissions, bid-ask spreads and market impact costs that affect the profitability of options trading. Realistically, transaction costs negatively impact on profit and therefore should not be ignored. By considering these transaction costs, the Black-Scholes-Merton model can provide a more accurate representation of the true cost and possible gains of trading options.

Ignoring transactions costs and assuming that volatility is constant can lead to inaccurate option pricing. It is therefore imperative to make adjustments in the Black-Scholes-Merton model by incorporating transaction cost and treating volatility as non-constant. This can help traders to make informed decisions by considering the impact of the transaction costs and non-constant volatility on their overall investment strategy and expected returns.

1.7 Significance of the Study

The results of this research will enable the potential investors to accurately compute possible return on investment based on accurate prediction of asset price and to minimize the risks associated with such investments. The study will also contribute to the knowledge in option price valuation in relation to Black-Scholes equation. The study will also bring about the impacts of non-constant volatility and transaction costs on option pricing which are of great concern to investors.The research findings will also be of significant to the government on policy formulation in controlling prices in stock exchange market.

1.8 Limitations of the study

This study considered a case of non-constant which varies only linearly but not quadratically nor cubically. Additionally, The study did not consider a case of three-dimensional Black-Scholes-Merton model equation. Further, the study did not show the stability and consistency of the Dufort-Frankel numerical finite difference scheme.

1.9 Assumptions

(i) There are no arbitrage opportunities

(ii) The underlying assets follow a lognormal random walk

(iii) An option is exercised only on expiration date

(iv) Correlation and free interest rates are kept constant

CHAPTER TWO

LITERATURE REVIEW

2.1 Introduction

This chapter undertakes a critical review of relevant literature and the contributions of various scholars in this area of study. Subsequently, the knowledge gap is identified.

2.2 Review of the related literature

Oh and Park (2023) studied GARCH option pricing with volatility derivatives. The study was based on benefits of joint estimations for GARCH option pricing that utilizes both stock returns and volatility derivatives. The estimated GARCH model yielded a persistent volatility component, which allows the leverage effect to hold up until long horizons. This persistent volatility component is crucial to modelling long term trail risk and pricing long term put options.

A study by Morales-Banuelos et. al. (2022) on a modified Black-Scholes-Merton model for Option Pricing, provided flexibilities for the markets. The study suggested that conformable Black-Scholes-Merton model may provide a way of valuing European call option compared to classical Black-Scholes model and Fractional Black-Scholes model. The study further showed that a more robust statistical procedure is required to improve the accuracy of option valuation. Additionally, the study suggested that an aspect of interest in asset pricing model to be considered is that of the transaction cost.

A study by DC et. al. (2022) on the solution of the Black-Scholes equation by finite difference schemes applied three finite difference methods that is, explicit, implicit and Crank-Nicolson to solve the Black-Scholes partial differential equation for the European call option. The study suggested that in future, there is need to find the numerical solution of Black-Scholes equation for European option pricing with non-constant volatility and interest rate.

Hossan et. al. (2022) investigated numerical solutions of the non-linear Black-Scholes partial differential equation which often appears in financial markets for European option pricing in the presence of transaction costs. The study exploited the transformations for the computational purpose of a nonlinear Black-Scholes partial differential equation to modify as a nonlinear parabolic type of partial differential equation with initial and boundary conditions for both call and put options. The study derived several schemes using Finite Volume Method and Finite Difference Method. The study established that both methods provide numerical solutions which are closer to exact solutions.

A study by Tseng et. al. (2021) on Lie algebraic approach for determining pricing for trade account options examined the options for the trade account using the Lie symmetry analysis. The study demonstrated that Lie symmetry technique can be used to analyze systemic problems in financial field although the method requires rigorous solutions to the numerous algebraic expressions.

Salvador and Oosterlee (2021) investigated total value adjustment for a stochastic volatility model and compared to Black-Scholes model. The study modified the Black-Scholes equation by introducing stochastic volatility. The resulting partial differential equation was solved by the method of characteristics for time discretization combined with finite element method. The modified model performed better than the Black-Scholes model.

A study by Njoroge (2020) on the Hybrid GARCH (1,1) European Option Pricing Model with Ensemble Empirical Mode Decomposition revealed that the proposed Hybrid GARCH (1,1) European Option Pricing Model with Ensemble Empirical Mode Decomposition model improves the price estimates of options better than BSM73 and GARCH-M (1,1). In addition, time to maturity of options greatly affect the accuracy of estimated option prices. Further, the study established that the proposed model hybrid GARCH (1,1) improves the approximation of option prices.

Chowdhury et. al. (2020) applied a comparative analytical approach and numerical technique to determine the price of call option and put option of an underlying asset in the frontier markets so as to predict stock price. The study modified the Black-Scholes model so as to determine parameters such as strike price and expiration time. Machine learning approach was applied using Rapidminer software. The approach showed better results over classical Black-Scholes Option Pricing model. The study further considered numerical calculation of volatility and established that as the price of stocks goes up due to overpricing, volatility also increases at a high rate. The study did not however considered other parameters affecting option value such as transaction costs.

A study by Ndede et. al. (2019) investigated the numerical effects of asset price fluctuation using BMPDE with relaxed assumptions. In this article, numerical method of Crank-Nicolson scheme was applied. The study showed that an increase in correlation coefficient on the two assets results to an increase in value of the call option value but a decrease in put option value. The study further revealed that the values for both call and put option decrease with an increase in dividend payout.

Liu et. al. (2019) carried out a study on option pricing under the Jump Diffusion and Multifactor Stochastic Processes. The study illustrated that the price of an option rises with the volatility value and jump rate which is in tandem with the real-life situation. The study also discovered that the price of an option reduces with the slow scale rate of volatility and goes up with the fast scale rate of volatility and the result of the fast scale volatility in the long run is lower than the effect of slow scale volatility. Additionally, variance swap of the strike price was also found to be negatively correlated with the time to maturity of an option. The study suggested that it is important to investigate the use of numerical methods so as to get more accurate solutions of high dimensional PDEs of option pricing.

Liu and Huang (2019) conducted a research on option pricing of carbon option based on GARCH and Black-Scholes model. The study combined GARCH and BSM as a way of improving on the constant volatility, an assumption in the Black-Scholes model. The study showed that combining the two models would improve the accuracy of carbon option pricing.

Aguilar and Korbel (2019) carried out a study on simple formulas for pricing and hedging European options in the Finite Moment Log-stable model. The study showed that as stability parameters increase, both call and put option also increase. The study however assumed that volatility is constant and no transaction cost is incurred in trading.

A study by Jankova (2018) outlined the limitations of Black-Scholes model for option pricing. The study demonstrated that the weakness of the model lies in the assumption that volatility is a constant, which is considerably not practical. The study also revealed that using stochastic and deterministic volatility would yield improved estimate of option contract price.

Nyakinda (2018) carried out a study on Logistic BSMPDE with stochastic volatility and its application on the prediction of prices of an asset was derived when volatility is varied rather than non-stochastic. The study recommended that the differential equation be solved so as to enable an accurate asset price prediction.

Anwar and Andallah (2018) studied numerical solution of Black-Scholes model. The study derived an explicit finite difference scheme and established a stability condition of the scheme. The relative error of the explicit scheme was also estimated by comparing the numerical solution with the analytical solution. The model equation solved explicitly did not factor in the non-constant volatility and the transaction cost.

Edeki et. al. (2017) analytically solved the time fractional order BSM for stock option valuation on a non-dividend yield basis. In this study, it was established that the rate of convergence of the acquired solution is faster than the exact form of solutions.

Mondal et. al. (2017) developed financial mathematical analysis model to obtain market options price with changing volatilities. The results that were obtained revealed that the approximations of option price using Heston model improves accuracy in pricing. However, the model has limitations in estimating pricing options which take a longer time to maturity and therefore yielding unrealistic prices. It also demonstrated that if the correlation coefficient is less than zero then the volatility would increase as the asset price decreases. On the other hand, if the correlation coefficient is greater than zero the volatility will increase leading to the decrease in the asset price return.

Rao and Manisha (2016) developed a higher order numerical scheme for pricing European call option governed by generalized Black-Scholes model.

The scheme was constructed in away that both time and space could be discretized simultaneously which leads to simpler convergence analysis. In the study, a two step backward diffrentiation formula was applied for temporal discretization and Higher-Order Difference approximation with Identity Expansion scheme with three nodal points for spatial discretization.

Krznaric (2016) compared the option price from Black-Scholes model to actual values. The study used a combination of historical data and generated normal distributions. The study first calculated and generated graphs on option prices through Black-Scholes formulas. With the help of software data analysis tool R, the real option prices were calculated using pricing formulas and the real data with no simulated numbers. The Black-Scholes method was compared to the real prices. The study revealed that the Black-Scholes model underpriced the options at the expiration date because of the unrealistic assumptions used. One of the assumptions used was that volatility is constant over time. This consequently failed to realize spikes in the stock data.

Agana et. al. (2016) carried out numerical investigation of the generalized BSM pricing in an illiquid market with transaction cost. The study showed that the existence of transaction costs, price slippage and the large traders in a financial market that is not perfectly liquid, impact heavily on options prices which have effects on the market volatility and a drift of the underlying asset and options prices. Moreover, it was revealed that an increase in the price of the underlying asset results to an increase in the value of the call option and a decrease in the value of the put option. The results also demonstrated that European options become more volatile due to a rise in transaction costs and the price impact from an illiquid market.

Meissner (2016) asserts that a rise in volatility raises the value of the option in correlation trading techniques.

Additionally, the study shown that when volatility increases, the value of the call option decreases when volatility increases and vice versa when volatility decreases and the correlation coefficient rises.

Mohammadi (2015) presented a numerical treatment for the generalized Black-Scholes partial differential equation arising from European option pricing using collocation method with the quantic B-spline functions. The study applied the quantic B-Spline collocation method in space and the Crank-Nicolson finite difference scheme in time. The study established that the proposed method is convergent and unconditionally stable.

Uddin et. al. (2013) carried out an investigation on a linear Black-Scholes model using a numerical solution. The numerical results of semi-discrete and full schemes for both European call and put options by Finite difference method and Finite element method were applied. The study established that as the price of the asset increases, the value of call option also increases. The study also compared the different methods of solving linear algebra such as Gauss-Seidel, Successive Over Relaxation method and Preconditioned Conjugate Gradients method which established that The Preconditioned Conjugate method proves to be the best method.

A study by Yousuf et. al. (2012) on the numerical approximation of nonlinear Black-Scholes model for exotic path-dependent American options with transaction cost employed a second order exponential time differencing method based on the Cox and Mathews approach. The study established that the method is more stable and efficient for solving nonlinear Black-Scholes model. Further, the study showed that the method does not incur unwanted oscillations unlike the exponential time differencing Crank-Nicolson method for exotic path-dependent American options.

Kumar et. al. (2012) studied analytical solution of fractional Black-Scholes European option pricing equation by using Laplace transform. The study combined the form of Laplace and the homotopy perturbation method to obtain a quick and accurate solution to the fractional Black-Scholes equation with boundary condition for a European option pricing problem. The proposed scheme found the solution without any discretization or restrictive assumptions and free from round off errors thereby reducing numerical computation to a greater extent.

Mitra (2012) studied an option pricing model that combines neural network approach and Black-Scholes formula. The study made an attempt to improve accuracy of the Black-Scholes model by applying artificial neural networks where input parameters are adjusted by a suitable multiplier. The study established that artificial neural networks are a better alternative.

Cen and Le (2011) presented a numerical method for generalized Black-Scholes equation for option pricing. The study employed central difference spatial discretization on a piecewise uniform mesh and an implicit time stepping technique. The study developed a matrix associated with discrete operator M-matrix which ensured a stable scheme. The scheme proved to be stable for arbitrary volatility and arbitrary interest rate.
The study further established that the scheme is second order convergent with respect to spatial variable.

Onyango et. al. (2010) carried out a study on the Walrasian-Samuelson Price Adjustment Model. In the study, an It$\hat{o}$ method for modeling the changes of the market value of securities traded due to new information which affects the market asset supply and demand was introduced. It is formulated on the basis of market supply, demand functions and the equilibrium price (Walrasian price) adjustment assumption. In this article, it is established that the proportional price increase is driven by excess demand.

The study established that if the supply and demand curves turned to be linear from the point of equilibrium, then the process changes to become logistic equation of Brownian motion with Wiener type of the random element.

Bakshi et. al. (2010) studied option pricing and hedging performance under stochastic volatility and stochastic interest rates. The study developed an implementable option model in closed form that admits both stochastic volatility and stochastic interest rates. The study compared the option prices when the volatility and interest rates are constant as in the Black-Scholes model with other cases when volatility is constant but interest rate is stochastic and when the volatility and interest rates are both stochastic. The study established that when volatility and interest rate are both stochastic then the option values become more accurate than when the volatility and interest rate are treated as constants.

Wang et. al. (2010) studied scaling and long range dependence in option pricing: Pricing European option with transaction cost under the mixed Brownian-fractional Brownian model. The study established that the minimal price of an option under transaction cost obtained showed that time step and Hurst exponent play an important role in option pricing. The study also showed that there exists fundamental difference between continuous time trade and discrete time trade and that continuous time trade assumption will result in underestimating the value of a European option.

Wong and Zhao (2010) investigated the valuation of currency options when the underlying currency follows a mean reverting lognormal process with multi scale stochastic volatility. A closed form solution was derived for the characteristic function of the log asset price. The European options were then valued by means of Fourier inversion formula. The study adopted the fractional fast Fourier transform to implement the Fourier inversion. The results were compared to Monte Carlo simulation. The study revealed that Fourier inversion formula performs better.

Company et. al. (2008) studied numerical solution of linear and non-linear Black-Scholes option pricing equations by means of semidiscretization technique. The study revealed that for a linear case, a fourth order discretization with respect to the underlying asset variables allows a better accurate approximation solution while for the non-linear case of interest modeling option pricing with transaction cost, semi discretization technique provides a competitive numerical solution. The study revealed that in practice, transaction costs arise when trading securities. The results of the study further demonstrated that although such transaction costs are generally small for institution investors, their influence lead to significant increase in the option price.

Amornwatttana et. al. (2007) constructed an option-pricing model for volatility estimating known as Hybrid Model. In this study, Neural Networks method is applied. The research demonstrated the advantages of using the option pricing of Neural Networks method. This model is applicable in the interpretation and understanding of the past data, prediction of volatility and accounting for conditions, which are not ideal and which make the BSM to have so many drawbacks. The results in this study demonstrated an improvement in performance of Black-Scholes model when a neural network for estimating volatility is combined with the hybrid neural network.

Doran and Ronn (2005) examined the extent of the bias between Black and Scholes and Black implied volatility. The study demonstrated that Black-Scholes is an upward-biased predictor of future realized volatility. The study applied methodology to options on energy contracts, a market in which crises are characterized by a positive correlation between price returns and volatilities. The study further established the bias in Black-Scholes and Black implied volatility to a negative market price of volatility risk.

Düring et. al. (2003) studied High order compact finite difference schemes for Nonlinear Black-Scholes equation. The study developed a nonlinear model with transaction cost arising in the hedging of portfolio which was discretized semi implicitly using high order compact finite difference schemes. The high order finite difference scheme proved to be conditionally stable and non-oscillatory.

Deb et. al. (2000) studied how to incorporate volatility and risk in electricity price forecasting. The study showed that traditional production costing models do not represent the multi commodity electricity market and also ignored both the transmission constraints and volatility. The study further revealed that ignoring volatility and transmission constraints makes it difficult to evaluate the emerging competitive electricity markets. The study provided a better method of analysis by models which simulate the volatility and causing electricity price swings with Multi commodity, Multi area Optimal Power Flow model.

Scott (1987) came up with model for options pricing which allows the variation of parameter to change randomly. The model applied a continuous time diffusion process that captures behaviour of stock return volatility. The study established that random variance model marginally improves the estimation of option prices.

2.3 Identification of Knowledge Gap

As illustrated in the literature review, many studies have been done on option pricing. In spite of this, inaccuracy in option pricing still remains a challenge.
This inaccuracy is attributed to constant volatility and zero transaction cost as demonstrated in Black-Scholes-Merton model. Based on these limitations, this research fills the knowledge gap by employing non-constant volatility and incorporating transaction cost with the objective of achieving more accurate price estimates of assets.

The accuracy in option pricing has been achieved by applying Dufort-Frankel numerical finite difference method for non-linear two-dimensional BSMPDE.

CHAPTER THREE

METHODOLOGY

3.1 Introduction

This chapter shows the derivation of the Black-Scholes-Merton Partial Differential Equation (BSMPDE) with the transaction cost and volatility. It also gives the finite difference scheme used and the discretization of the Black-Scholes-Merton partial differential model equation.

3.2 The original Black-Scholes-Merton equation

The original One -dimensional Black-Schole-Merton equation (Black and Scholes 1973) is given by;

$$\frac{\partial V}{\partial t} + r\frac{\partial V}{\partial S}S + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0 \tag{3.1}$$

Where; V is the value of the option, t is the time before expiration, r is the risk free interest rate, S is the first asset price and σ is the volatility of the first asset.
The equation (3.1) can be modified to form a Two-dimensional Black-Scholes-Merton Partial Differential equation given by;

$$\frac{\partial V}{\partial t} + r\left[\frac{\partial V}{\partial S_1}S_1 + \frac{\partial V}{\partial S_2}S_2\right] + \frac{1}{2}\left[\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + \sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2}\right] + \rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} - rV = 0 \tag{3.2}$$

Where; V is the value of the option, t is the time before expiration,
r is the risk free interest rate, S_1 is the first asset price, S_2 is the second asset price, σ_1 is the volatility of the first asset, σ_2 is the volatility of the second asset and ρ is the coefficient of the two assets (Ndede et. al. 2019).

3.3 Deriving the BSMPDE

Equation (3.2) is derived as demostrated:
Consider the Geometric Brownian motion which describes price evolution of an asset as a sum of the rate of growth of asset price with time and random change in the asset price given by;

$$dS = \mu S dt + \sigma S dz \tag{3.3}$$

Where, dz is a Wiener process. In order to model an asset price distribution correctly in a lognormal fashion, a stochastic version of the chain rule will be used to solve a stochastic differential equation representing Brownian motion. The ordinary calculus version of the chain rule can be extended to agree with the random variables.

3.3.1 The Taylor series;

Suppose $f(x)$ is analytic about $x = a$, The Taylor series about $x = a$ is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n \tag{3.4}$$

Expanding the Taylor series gives;
$f(x) = f(a) + \sum_{n=1}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$
Which gives,

$$f(x) - f(a) = f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ... \tag{3.5}$$

Now replacing x with $(x+\Delta x)$ and then replacing a with x yields;

$$f(x+\Delta x) - f(x) = f'(x)(\Delta x) + \frac{f''(x)}{2!}(\Delta x)^2 + \cdots \tag{3.6}$$

Denoting

$f(x+\Delta x) - f(x)$ by Δf in (3.5)

we get;

$$\Delta f = f'(x)(\Delta x) + \frac{f''(x)}{2!}(\Delta x)^2 + \ldots \tag{3.7}$$

Taking the relation between a small change in f, (Δf) to be denoted by df and a small change in x, (Δx) to be denoted by dx and assuming that the small change in x is very small such that the terms of order ≥ 2 can be neglected, then equation (3.6) reduces to the simpler form

$$df = f'dx + \frac{f''(x)}{2!}(dx)^2 + \ldots \tag{3.8}$$

The Taylor series in equation (3.8) helps us in deriving the It$\hat{o}$'s lemma.
The It$\hat{o}$'s lemma is an essential component of It$\hat{o}$ calculus. It is important in deriving differential equations for the value of the derivative securities such as stock options. It is also used to determine the value of a time dependent function of a stochastic process.

3.3.2 It$\hat{o}$'s lemma;

Let dz be a Wiener process and $S(t)$ be an It$\hat{o}$ process which satisfies the Stochastic differential equation;

$$dS = \mu(S,t)dt + \sigma(S,t)dz \tag{3.9}$$

Where μ is the drift coefficient and σ is the volatility.

If $V = V(S,t)$ is a twice continuously differentiable function for second partial derivatives of V are continuous functions then $V(S(t),t)$ is also an It$\hat{o}$'s process and its total differential is given by;

$$dV = \left[\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial S} + \frac{1}{2}\mu^2 S^2\frac{\partial^2 V}{\partial S^2}\right]dt + \sigma S\frac{\partial V}{\partial S}dz \tag{3.10}$$

Proof

Let $V(S(t),t)$ be a function which depends on both, t and $S(t)$ i.e $V(S(t),t)$ where t represents a time variable and $S(t)$ represents a Stochastic process such Brownian motion. From the Taylor's theorem, the total differential of V is given by

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS)^2 + \frac{1}{2}\frac{\partial^2 V}{\partial t^2}(dt)^2 + \frac{\partial^2 V}{\partial t \partial S}dtdS \tag{3.11}$$

Since $dS = \mu S dt + \sigma S dz$ as shown in equation (3.9),

on squaring both sides of equation (3.9), we obtain,

$$(dS)^2 = \mu^2 S^2(dt)^2 + 2\mu\sigma S^2 dtdz + \sigma^2 S^2(dz)^2 \tag{3.12}$$

Writing $dz_i + dt - z_i$ then by definition, dz_i is normally distributed with mean equal to 0 and variance equal to dt, that is, $dz_i \sim N(0, dt)$. Computing the expected value of $(dz_i)^2$ gives

$$E\left[(dz_i)^2\right] = \text{Var}(dz_i) - (E[d_i])^2 = dt - 0^2 = dt \tag{3.13}$$

Hence we can say that $(dz_i)^2 = dt$ i.e $(dz)^2 = (dt)$ thus we have $(dz)^2 = (dt)$ that is, $dtdz = (dt)^{\frac{3}{2}}$ Given that $(dz)^2 = (dt)$ it implies that $dz = (dt)^{\frac{1}{2}}$ thus $dtdz = (dt)^{\frac{1}{2}}(dt) = (dt)^{\frac{3}{2}}$ hence upon substituting in equation (3.12) we obtain,

$$(dS)^2 = \mu^2 S^2 (dt)^2 + 2\mu\sigma S^2 (dt)^{3/2} + \mu^2 S^2 dt \tag{3.14}$$

By substituting equation (3.14) in equation (3.11) and neglecting all the terms involving dt , set $(dt)^{3/2} = 0$, it follows that $(dt)^2 = 0$, we obtain a simplified equation;

$$dV = \left[\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2}\mu^2 S^2 \frac{\partial^2 V}{\partial S^2}\right] dt + \sigma S \frac{\partial V}{\partial S} dz \tag{3.15}$$

Therefore equation (3.15) is the It$\hat{o}$'s lemma for one variable, in this case one asset S .

3.3.3 Two-dimensional BSMPDE

Using the Geometric Brownian motion with two different assets S_1 and S_2 , the Taylor series and It$\hat{o}$ lemma, we proceed and derive the two-dimensional BSMPDE as follows; Considering the European option whose pay off depends on the prices of the two assets and , we have;

$$dS_1 = \mu_1 S_1 dt + \sigma_1 S_1 dz_1$$

$$dS_2 = \mu_2 S_2 dt + \sigma_2 S_2 dz_2 \tag{3.16}$$

Taking the random numbers dS_1 and dS_2 and to be correlated such that $E\left[dS_1, dS_2\right] = \rho dt$ and constructing the portfolio π for the two assets, we have

$$\pi = V - \Delta_1 dS_1 - \Delta_2 dS_2 \tag{3.17}$$

The change on the value of the portfolio from t to dt is given by

$$d\pi = dV - \Delta_1 dS_1 - \Delta_2 dS_2 \tag{3.18}$$

From It$\hat{o}$'s lemma with two variables, we get

$$dV = \left[\frac{dV}{dt} + \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S^2} + \rho \sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} + \frac{1}{2}\sigma_1^1 S_2^2 \frac{\partial^2 V}{\partial S_2^2}\right] + \frac{\partial V}{\partial S_1} dS_1 + \frac{\partial V}{\partial S_2} dS_2 \tag{3.19}$$

Letting $\Delta_1 = \frac{\partial V}{\partial S_1}$ and $\Delta_2 = \frac{\partial V}{\partial S_2}$ then substituting, the portfolio (π) changes by the amount

$$d\pi = \left[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + \rho \sigma_1 \sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2}\right] dt \tag{3.20}$$

This is a risk-less change and is therefore equivalent to amount of cash in a risk-free interest bearing account;

$$d\pi = r\pi dt = r\left[V - \frac{\partial V}{\partial S_1} S_1 - \frac{\partial V}{\partial S_2} S_2\right] dt \tag{3.21}$$

Equating (3.20) and (3.21) we get

$$\frac{\partial V}{\partial t}+r\left[\frac{\partial V}{\partial S_1}S_1+\frac{\partial V}{\partial S_2}S_2\right]+\frac{1}{2}\left[\sigma_1^2S_1^2\frac{\partial^2 V}{\partial S_1^2}+\sigma_2^2S_2^2\frac{\partial^2 V}{\partial S_2^2}\right]+\rho\sigma_1\sigma_2S_1S_2\frac{\partial^2 V}{\partial S_1\partial S_2}-rV=0 \tag{3.22}$$

Thus equation (3.22) is the two-dimensional BSMPDE.

3.4 Transaction Costs

Assume the two assets S_1 and S_2 considers the transaction costs C_1 and C_2 meaning that at a time dt each of the assets incurs a transaction cost C_1S_1dt and C_2S_2dt respectively, letting σ to represent the volatility and r to represent the risk free rate then adding the transaction costs on each of the underlying assets from equation (3.22) we obtain

$$\frac{\partial V}{\partial t}+r\left[\frac{\partial V}{\partial S_1}S_1+\frac{\partial V}{\partial S_2}S_2\right]-\left[C_1S_1\frac{\partial V}{\partial S_1}+C_2S_2\frac{\partial V}{\partial S_2}\right]+\frac{1}{2}\left[\sigma_1^2S_1^2\frac{\partial^2 V}{\partial S_1^2}+\sigma_2^2S_2^2\frac{\partial^2 V}{\partial S_2^2}\right]+$$
$$\rho\sigma_1\sigma_2S_1S_2\frac{\partial^2 V}{\partial S_1\partial S_2}-rV=0 \tag{3.23}$$

Thus, equation (3.23) represents the standard two-dimensional BSMPDE considering the transaction costs C , volatility σ and correlation between the two assets ρ . It forms the governing equation of study.

Using the transformations of independent variables (Ndede et. al. 2019) for example transforming from $[S_1,S_2]$ to $[x,y]$ where;
$x=\ln(S_1)-\left(r-\frac{1}{2}\delta_1^2\right)t$ and $y=\ln(S_2)-\left(r-\frac{1}{2}\delta_2^2\right)t$ and taking the derivatives of x and y with respect to S_1 and S_2, it will be transformed to:

$$\frac{\partial x}{\partial S_1}=\frac{1}{S_1},\frac{\partial x}{\partial S_2}=0,\frac{\partial y}{\partial S_1}=0,\frac{\partial y}{\partial S_2}=\frac{1}{S_2},\frac{\partial t}{\partial S_1}=0,\frac{\partial t}{\partial S_2}=0 \tag{3.24}$$

Transforming the PDEs using the chain rule and replacing equation (3.24) into it from the 2D BSMPDE to obtain;

$$\frac{\partial V}{\partial S_1} = \frac{\partial V}{\partial x}\frac{\partial x}{\partial S_1} + \frac{\partial V}{\partial y}\frac{\partial y}{\partial S_1} + \frac{\partial V}{\partial t}\frac{\partial t}{\partial S_1} = \frac{1}{S_1}\frac{\partial V}{\partial x} \tag{3.25}$$

$$\frac{\partial V}{\partial S_2} = \frac{\partial V}{\partial x}\frac{\partial x}{\partial S_2} + \frac{\partial V}{\partial y}\frac{\partial y}{\partial S_2} + \frac{\partial V}{\partial t}\frac{\partial t}{\partial S_2} = \frac{1}{S_2}\frac{\partial V}{\partial y} \tag{3.26}$$

$$\frac{\partial^2 V}{\partial S_1^2} = \frac{\partial}{\partial S_1}\left[\frac{\partial V}{\partial S_1}\right] = \frac{\partial}{\partial S_1}\left[\frac{1}{S_1}\frac{\partial V}{\partial x}\right] = \frac{1}{S_1}\frac{\partial}{\partial S_1}\left[\frac{\partial V}{\partial x}\right] = \frac{1}{S_1}\frac{\partial}{\partial x}\left[\frac{1}{S_1}\frac{\partial V}{\partial x}\right] = \frac{1}{S_1^2}\frac{\partial^2 V}{\partial x^2} \tag{3.27}$$

$$\frac{\partial^2 V}{\partial S_2^2} = \frac{\partial}{\partial S_2}\left[\frac{\partial V}{\partial S_2}\right] = \frac{\partial}{\partial S_2}\left[\frac{1}{S_2}\frac{\partial V}{\partial y}\right] = \frac{1}{S_2}\frac{\partial}{\partial S_2}\left[\frac{\partial V}{\partial y}\right] = \frac{1}{S_2}\frac{\partial}{\partial y}\left[\frac{1}{S_2}\frac{\partial V}{\partial y}\right] = \frac{1}{S_2^2}\frac{\partial^2 V}{\partial y^2} \tag{3.28}$$

$$\frac{\partial^2 V}{\partial S_1 \partial S_2} = \frac{\partial}{\partial S_1}\left[\frac{\partial V}{\partial S_2}\right] = \frac{1}{\partial S_1}\left[\frac{1}{S_2}\frac{\partial V}{\partial y}\right] = \frac{1}{S_2}\frac{\partial}{\partial S_1}\left[\frac{\partial V}{\partial y}\right] = \frac{1}{S_2}\frac{\partial}{\partial y}\left[\frac{\partial V}{\partial S_1}\right] = \frac{1}{S_2}\frac{\partial}{\partial y}\left[\frac{1}{S_1}\frac{\partial V}{\partial x}\right] = \frac{1}{S_1 S_2} \tag{3.29}$$

Substituting the derivatives above into equation (3.23) and assuming $C_1 = C_2 = C$ leads to:

$$\frac{\partial V}{\partial t} + (r - C)\left[\frac{\partial V}{\partial x} + \frac{\partial V}{\partial y}\right] + \frac{1}{2}\left[\sigma_1^2 \frac{\partial^2 V}{\partial x^2} + \sigma_2^2 \frac{\partial^2 V}{\partial y^2}\right] + \rho \sigma_1 \sigma_2 \frac{\partial^2 V}{\partial x \partial y} - rV = 0 \tag{3.30}$$

Thus, equation (3.30) becomes the 2D BSMPDE with non-constant volatility σ and transaction cost C which is to be discretized and numerically solved.

3.5 Dufort-Frankel Difference Scheme

The Dufort-Frankel scheme, proposed in 1953 (Dufort and Frankel, 1953), makes use of various themes discussed in Richardson and Modified Alternating Direction Explicit(MADE) schemes.

It came into existence in an effort to address the instability associated with the Richardson scheme (Smith, 1984). The scheme is explicit, unconditionally stable and second order accurate in both the spatial and temporal dimensions. The Dufort-Frankel scheme is conditionally consistent with the partial differential equation it solves. Furthermore, since the Dufort-Frankel scheme is a two step method, calculating the first temporal vector after the initial boundary requires some other method. The Dufort-Frankel scheme makes use of a time derivative estimation similar to the Richardson scheme. The Dufort-Frankel scheme is unconditionally stable. This result is well documented and often shown for pure diffusion parabolic partial differential equations (Smith, 1984).

3.6 Discretization of Black-Scholes-Merton equation

Discretization of equation (3.30) is done as follows where V_t is forward difference, and V_{xx}, V_{yy} and V_{xy} are central difference scheme but the value of $V_{i,j}^n$ in V_{xx} and V_{yy} are replaced by $\left(V_{i,j}^{n+1} + V_{i,j}^{n-1}\right)$ difference approximation.
The finite difference approximations for the partial derivatives appearing in (3.30) are given by,

$$\frac{\partial V}{\partial t} = \frac{V_{i,j}^{n+1} - V_{i,j}^{n-1}}{2\Delta t} \tag{3.31}$$

$$\frac{\partial^2 V}{\partial x^2} = \frac{V_{i+1,j}^n - \left(V_{i,j}^{n+1} + V_{i,j}^{n-1}\right) + V_{i-1,j}^n}{(\Delta x)^2} \tag{3.32}$$

$$\frac{\partial^2 V}{\partial y^2} = \frac{V_{i,j+1}^n - \left(V_{i,j}^{n+1} + V_{i,j}^{n-1}\right) + V_{i,j-1}^n}{(\Delta y)^2} \tag{3.33}$$

$$\frac{\partial^2 V}{\partial x \partial y} = \frac{V_{i+1,j+1}^n - V_{i-1,j+1}^n - V_{i+1,j-1}^n + V_{i-1,j-1}^n}{4(\Delta x)(\Delta y)} \tag{3.34}$$

3.7 Dufort-Frankel numerical scheme

In this section, we discretize the Black-Scholes-Merton option pricing partial differential equation (3.30) and form a Dufort- Frankel numerical scheme which we eventually solve using the finite difference method. Equation (3.30) is discretized to study the effects of C, and σ for call and put option value. In the Dufort-Frankel numerical scheme, V_t, V_{xx} , V_{yy} and V_{xy} are replaced by the central finite approximations. But the value of $V_{i,j}^n$ in V_{xx} and V_{yy} are replaced by $\left(V_{i,j}^{n+1}+V_{i,j}^{n-1}\right)$ difference approximation as seen in (3.31), (3.32), (3.33) and (3.34). When these approximations are substituted into equation (3.30), and let $r=\rho=1$ and $\sigma_1=\sigma_2=\sigma$, we get

$$\frac{V_{i,j}^{n+1}-V_{i,j}^{n-1}}{2\Delta t}+(1-C)\left[\frac{V_{i+1,j}^n-V_{i-1,j}^n}{2\Delta x}+\frac{V_{i,j+1}^n-V_{i,j-1}^n}{2\Delta y}\right]+$$
$$\frac{\sigma}{2}\left[\frac{V_{i+j,j}^n-\left(V_{i,j}^{n+1}+V_{i,j}^{n-1}\right)+V_{i-j}^n}{(\Delta x)^2}\right]+\frac{\sigma}{2}\left[\frac{V_{i,+1}^n-\left(V_{i,j}^{n-1}+V_{i,j}^{n-1}\right)+V_{i,j-1}^n}{(\Delta y)^2}\right]$$

$$+\sigma^2\left[\frac{V_{i+1,+1}^n-2V_{i-1,+1}^n-V_{i+1j+1}^n+V_{i-1,-1}^n}{4\Delta x)(\Delta y)}\right]-V_{i,j}^n=0 \qquad (3.35)$$

Taking $\phi=\frac{\Delta t}{(\Delta x)^2}=\frac{\Delta t}{(\Delta y)^2}, \varphi=\frac{\Delta t}{\Delta x}=\frac{\Delta t}{\Delta y}, \Delta x=\Delta y$ on a square mesh and multiplying all the terms in equation (3.36) by $4\Delta t$ and re-arranging, we obtain the scheme;

$$(2+2\phi\sigma)V_{i,j}^{n+1}-4\Delta t V_{i,j}^n+(2\varphi(r-c)+2\phi\sigma \quad V_{i+1,j}^n+(2\varphi(r-c)+2\phi\sigma)$$
$$V_{i-1,j}^n+(-2-4\phi\sigma)V_{i,j}^{n-1} \quad +(2\varphi(r-c)+2\phi\sigma)V_{i,j-1}^n-\phi\rho\sigma^2V_{i+1,j+1}^n \quad +$$
$$\phi\rho\sigma^2\left(V_{i+1,j+1}^n-2V_{i-1,j+1}^n-V_{i+1,j-1}^n+V_{i-1,j-1}^n\right)=0$$

$$(3.36)$$

Taking , $\Delta x=\Delta y=0.1$ and $\Delta t=0.01, \Rightarrow \phi=1, \varphi=0.1$ and multiply by 10 in (3.37) the Du Fort-Frankel scheme is obtained

$$(2-4\sigma)V_{i,j}^{n+1} = 0.04V_{i,j}^n-(0.2-0.2C+2\sigma)V_{i+1,j}^n-(0.2-0.2C+2\sigma)V_{i-1,j}^n+(2+$$
$$2\sigma)V_{i,j}^{n-1}-(0.2-0.2C+2\sigma)V_{i,j+1}^n-(0.2+0.2C+2\sigma)V_{i,j-1}^n-\sigma^2V_{i+1,j+1}^n+2\sigma^2V_{i-1,j+1}^n+$$
$$\sigma^2V_{i+1,j-1}^n-\sigma^2V_{i-1,j-1}^n$$

$$(3.37)$$

Upon simplifying and dividing (3.37) by $2-4\sigma$, we get

$$V_{i,j}^{n+1}=\frac{1}{51-2\sigma)}V_{i,j}^{n}-\frac{(1-C+10\sigma)}{(10-20\sigma)}\left(V_{i+1,j}^{n}+V_{i-1,j}^{n}+V_{i,j+1}^{n}+V_{i,j-1}^{n}\right)+\frac{(1+\sigma)}{(1-2\sigma)}V_{ij}^{n-1}+$$

$$\frac{\sigma^2}{2(-2\sigma)}\left(V_{i+1,j-1}^{n}+2V_{i-1,j+1}^{n}-V_{i+1,j+1}^{n}-V_{i-1,j-1}^{n}\right) \tag{3.38}$$

Taking $n=1, i=1,2.\ldots.6$, j =1, i.e. $S_1=S_2$ the scheme in equation (3.38) can be written in algebraic equations as

$$V_{1,1}^{2}=-\frac{(1-C+10\sigma)}{(10-20\sigma}V_{2,l}^{1}+\frac{1}{5(1-2\sigma)}V_{1,1}^{1}-\frac{(1-C+10\sigma)}{(10-20\sigma)}V_{0,1}^{1}\frac{(1-C+10\sigma)}{(10-20\sigma)}\left(V_{1,2}^{1}+V_{1,0}^{1}\right)+\frac{(1+\sigma)}{(1-2\sigma)}V_{1,1}^{0}+$$

$$\frac{\sigma^2}{2(1-2\sigma)}\left(V_{2,0}^{1}+2V_{0,2}^{1}-V_{2,2}^{1}-V_{1,0}^{1}\right)$$

$$V_{2,1}^{2}=-\frac{(1-\alpha10\sigma)}{(10-20\sigma)}V_{3,1}^{1}+\frac{1}{5(1-2\sigma)}V_{2,1}^{1}-\frac{(1-C+10\sigma)}{(10-20\sigma)}V_{1,1}^{1}-\frac{(1-C+10\sigma)}{(10-20\sigma)}\left(V_{2,2}^{1}+V_{2,0}^{1}\right)+\frac{(1+\sigma)}{(1-2\sigma)}V_{2,1}^{0}+$$

$$\frac{\sigma^2}{2(1-2\sigma)}\left(V_{3,0}^{1}+2V_{1,2}^{1}-V_{3,2}^{1}-V_{2,0}^{1}\right)$$

$$V_{3,1}^{2}=-\frac{(1-C+10\sigma)}{(10-20\sigma)}V_{4,1}^{1}+\frac{1}{5(1-2\sigma)}V_{3,1}^{1}-\frac{(1-C+10\sigma)}{(10-20\sigma)}V_{2,1}^{1}-\frac{(1-C+10\sigma)}{(10-20\sigma}\left(V_{3,2}^{1}+V_{3,0}^{1}\right)+\frac{(1+\sigma)}{(1-2\sigma)}V_{3,1}^{0}+$$

$$\frac{\sigma^2}{2(1-2\sigma}\left(V_{4,0}^{1}+2V_{2,2}^{1}-V_{4,2}^{1}-V_{3,0}^{1}\right)$$

$$V_{4,1}^{2}=-\frac{(1-c+10\sigma}{(10-20\sigma}V_{5,1}^{1}+\frac{1}{5(1-2\sigma)}V_{4,1}^{1}-\frac{(1-C+10\sigma}{(10-20\sigma)}V_{3,1}^{1}-\frac{(1-C+10\sigma)}{(10-20\sigma)}\left(V_{4,2}^{1}+V_{4,0}^{1}\right)+\frac{(1+\sigma)}{(1-2\sigma)}V_{4,1}^{0}+$$

$$\frac{\sigma^2}{2(1-2\sigma)}\left(V_{5,0}^{1}+2V_{3,2}^{1}-V_{5,2}^{0}-V_{4,0}^{0}\right)$$

$$V_{5,1}^{2}=-\frac{1-C+10\sigma}{10-20\sigma}V_{6,1}^{1}+\frac{1}{5(1-2\sigma}V_{5,1}^{1}-\frac{(1-C+10\sigma}{10-20\sigma}V_{4,1}^{1}-\frac{(1-C+10\sigma}{(10-20\sigma}\left(V_{5,2}^{1}+V_{5,0}^{1}\right)+\frac{(1+\sigma}{(1-2\sigma)}V_{5,}^{0}+$$

$$\frac{\sigma^2}{2(1-2\sigma)}\left(V_{6,0}^{1}+2V_{4,2}^{1}-V_{6,2}^{1}-V_{5,0}^{1}\right)$$

$$V_{6,1}^{2}=-\frac{1-C+10\sigma}{10-20\sigma}V_{7,1}^{1}+\frac{1}{5(1-2\sigma}V_{6,1}^{1}-\frac{(1-C+10\sigma}{10-20\sigma}V_{5,1}^{1}-\frac{(1-C+10\sigma}{(10-20\sigma}\left(V_{6,2}^{1}+V_{6,0}^{1}\right)+\frac{(1+\sigma}{(1-2\sigma)}V_{6,}^{0}+$$

$$\frac{\sigma^2}{2(1-2\sigma)}\left(V_{7,0}^{1}+2V_{5,2}^{1}-V_{7,2}^{1}-V_{6,0}^{1}\right)$$

(3.39)

The above 6 algebraic equations can be written in matrix form as

$$\begin{bmatrix} V^2_{1,1} \\ V^2_{2,1} \\ V^2_{3,1} \\ V^2_{4,1} \\ V^2_{5,1} \\ V^2_{6,1} \end{bmatrix} = \begin{bmatrix} \frac{1}{5(1-2\sigma)} & -\frac{(1-C+10\sigma)}{(10-20\sigma)} & 0 & 0 & 0 & 0 \\ -\frac{1-C+10\sigma)}{(10-20\sigma)} & \frac{1}{5(1-2\sigma)} & -\frac{(1-C+10\sigma)}{(10-20\sigma)} & 0 & 0 & 0 \\ 0 & -\frac{(1-C+10\sigma)}{(10-20\sigma)} & \frac{1}{5(1-2\sigma)} & -\frac{(1-C+10\sigma)}{(10-20\sigma)} & 0 & 0 \\ 0 & 0 & -\frac{(1-C+10\sigma)}{(10-20\sigma)} & \frac{1}{5(1-2\sigma)} & -\frac{(1-C+10\sigma)}{(10-20\sigma)} & 0 \\ 0 & 0 & 0 & -\frac{(1-C+10\sigma)}{(10-20\sigma)} & \frac{1}{5(1-2\sigma)} & -\frac{(1-C+10\sigma)}{(10-20\sigma)} \\ 0 & 0 & 0 & 0 & -\frac{(1-C+10\sigma)}{(10-20\sigma)} & \frac{1}{5(1-2\sigma)} \end{bmatrix}$$

$$\begin{bmatrix} V^1_{1,1} \\ V^1_{2,1} \\ V^1_{3,1} \\ V^1_{4,1} \\ V^1_{5,1} \\ V^1_{6,1} \end{bmatrix} + \begin{bmatrix} -\frac{1-C+10\sigma}{10-20\sigma}\left(V^0_{1,2}+V^0_{1,0}\right) + \frac{(1+\sigma)}{1-2\sigma}V^{-1}_{1,1} + \frac{\sigma^2}{2(1-2\sigma)}\left(V^0_{2,0}+2V^0_{0,2}-V^0_{2,2}-V^0_{1,0}\right) \\ -\frac{1-C+10\sigma}{10-20\sigma}\left(V^0_{2,2}+V^0_{2,0}\right) + \frac{(1+\sigma)}{1-2\sigma)}V^{-1}_{2,1} + \frac{\sigma^2}{2(1-2\sigma)}\left(V^0_{3,0}+2V^0_{1,2}-V^0_{3,2}-V^0_{2,0}\right) \\ -\frac{1-C+10\sigma}{10-20\sigma)}\left(V^0_{3,2}+V^0_{3,0}\right) + \frac{(1+\sigma)}{(1-2\sigma)}V^{1}_{3,1} + \frac{\sigma^2}{2(1-2\sigma)}\left(V^0_{4,0}+2V^0_{2,2}-V^0_{4,2}-V^0_{3,0}\right) \\ -\frac{1-C+10\sigma}{10-20\sigma}\left(V^0_{4,2}+V^0_{4,0}\right) + \frac{(1+\sigma)}{(1-2\sigma)}V^{-1}_{4,1} + \frac{\sigma^2}{2(1-2\sigma)}\left(V^0_{5,0}+2V^0_{3,2}-V^0_{5,2}-V^0_{4,0}\right) \\ -\frac{1-C+10\sigma}{10-20\sigma}\left(V^0_{5,2}+V^0_{5,0}\right) + \frac{(1+\sigma)}{(1-2\sigma)}V^{-1}_{5,1} + \frac{\sigma^2}{2(1-2\sigma)}\left(V^0_{6,0}+2V^0_{4,2}-V^0_{6,2}-V^0_{5,0}\right) \\ -\frac{1-C+10\sigma}{10-20\sigma}\left(V^0_{6,2}+V^0_{6,0}\right) + \frac{(1+\sigma)}{1-2\sigma)}V^{-1}_{6,1} + \frac{\sigma^2}{2(1-2\sigma)}\left(V^0_{2,0}+2V^0_{5,2}-V^0_{7,2}-V^0_{6,0}\right) \\ \\ \end{bmatrix} \tag{3.40}$$

We use the initial conditions below in (3.40)

$$V(x,y,0) = 0, t = 0, x = y \tag{3.41}$$

Let the boundary conditions for call asset option be

$$V(x,0,t) = V(x,2,t) = 0, V(x,1,1) = e^x, t > 0, x \geq y \tag{3.42}$$

and the boundary conditions for put asset option be

$$V(x,0,t) = V(x,2,t) = 0, V(x,1,1) = e^{-x}, t > 0, x \geq y \tag{3.43}$$

Therefore for call option, we apply the conditions in (3.42) with $x = i$ in equation (3.40) to get the matrix

$$\begin{bmatrix} V_{1,1}^2 \\ V_{2,1}^2 \\ V_{3,1}^2 \\ V_{4,1}^2 \\ V_{5,1}^2 \\ V_{6,1}^2 \end{bmatrix} = \begin{bmatrix} \frac{1}{5(1-2\sigma)} & \frac{(1-C+10\sigma)}{(10-20\sigma)} & 0 & 0 & 0 & 0 \\ \frac{1-C+10\sigma)}{(10-20\sigma)} & \frac{1}{5(1-2\sigma)} & \frac{(1-C+10\sigma)}{(10-20\sigma)} & 0 & 0 & 0 \\ 0 & \frac{(1-C+10\sigma)}{(10-20\sigma)} & \frac{1}{5(1-2\sigma)} & \frac{(1-C+10\sigma)}{(10-20\sigma)} & 0 & 0 \\ 0 & 0 & \frac{(1-C+10\sigma)}{(10-20\sigma)} & \frac{1}{5(1-2\sigma)} & \frac{(1-C+10\sigma)}{(10-20\sigma)} & 0 \\ 0 & 0 & 0 & \frac{(1-C+10\sigma)}{(10-20\sigma)} & \frac{1}{5(1-2\sigma)} & \frac{(1-C+10\sigma)}{(10-20\sigma)} \\ 0 & 0 & 0 & 0 & \frac{(1-C+10\sigma)}{(10-20\sigma)} & \frac{1}{5(1-2\sigma)} \end{bmatrix} \begin{bmatrix} 2.71828 \\ 7.38906 \\ 20.0855 \\ 54.59518 \\ 148.41316 \\ 403.42879 \end{bmatrix} \tag{3.44}$$

Similarly, for put option we apply the conditions in (3.43) with $x = -i$ in (3.40) to obtain,

$$\begin{bmatrix} V_{1,1}^2 \\ V_{2,1}^2 \\ V_{3,1}^2 \\ V_{4,1}^2 \\ V_{5,1}^2 \\ V_{6,1}^2 \end{bmatrix} = \begin{bmatrix} \frac{1}{5(1-2\sigma)} & \frac{(1-C+10\sigma)}{(10-20\sigma)} & 0 & 0 & 0 & 0 \\ \frac{1-C+10\sigma)}{(10-20\sigma)} & \frac{1}{5(1-2\sigma)} & \frac{(1-C+10\sigma)}{(10-20\sigma)} & 0 & 0 & 0 \\ 0 & \frac{(1-C+10\sigma)}{(10-20\sigma)} & \frac{1}{5(1-2\sigma)} & \frac{(1-C+10\sigma)}{(10-20\sigma)} & 0 & 0 \\ 0 & 0 & \frac{(1-C+10\sigma)}{(10-20\sigma)} & \frac{1}{5(1-2\sigma)} & \frac{(1-C+10\sigma)}{(10-20\sigma)} & 0 \\ 0 & 0 & 0 & \frac{(1-C+10\sigma)}{(10-20\sigma)} & \frac{1}{5(1-2\sigma)} & \frac{(1-C+10\sigma)}{(10-20\sigma)} \\ 0 & 0 & 0 & 0 & \frac{(1-C+10\sigma)}{(10-20\sigma)} & \frac{1}{5(1-2\sigma)} \end{bmatrix} \begin{bmatrix} 0.36789 \\ 0.13533 \\ 0.04978 \\ 0.018316 \\ 0.006738 \\ 0.002479 \end{bmatrix} \tag{3.45}$$

Solving the matrices in (3.44) and (3.45), the results for effects of σ and C for call and put option values are obtained and presented in tables and graphs in chapter four.

CHAPTER FOUR

RESULTS AND DISCUSSION

4.1 Introduction

This chapter shows the results obtained from the numerical simulations with the use MATLAB computer program. The simulation results given focus on the effects of volatility σ, and transaction cost C, on call and put asset option values.

4.2 Effects of volatility on call option value

Equation (3.44) is solved using MATLAB and the results for the effects of varying volatility on call option value are presented as shown in table 4.1 below

Table 4.1
Call option values for varying volatility

Volatility	Asset Price , S($)					
	50	60	70	80	90	100
σ = 0.1	1.0190	1.1285	1.2910	2.2424	4.3754	1.5989
σ = 0.3	1.0615	1.1607	1.4759	2.4569	4.9743	1.9975
σ = 0.5	1.0678	1.2409	1.5553	2.7819	5.92807	2.13460
σ = 0.7	1.0931	1.28769	1.779731	3.119986	6.762397	2.953109

The above results in Table 4.1 are presented in Figure 4.1 below

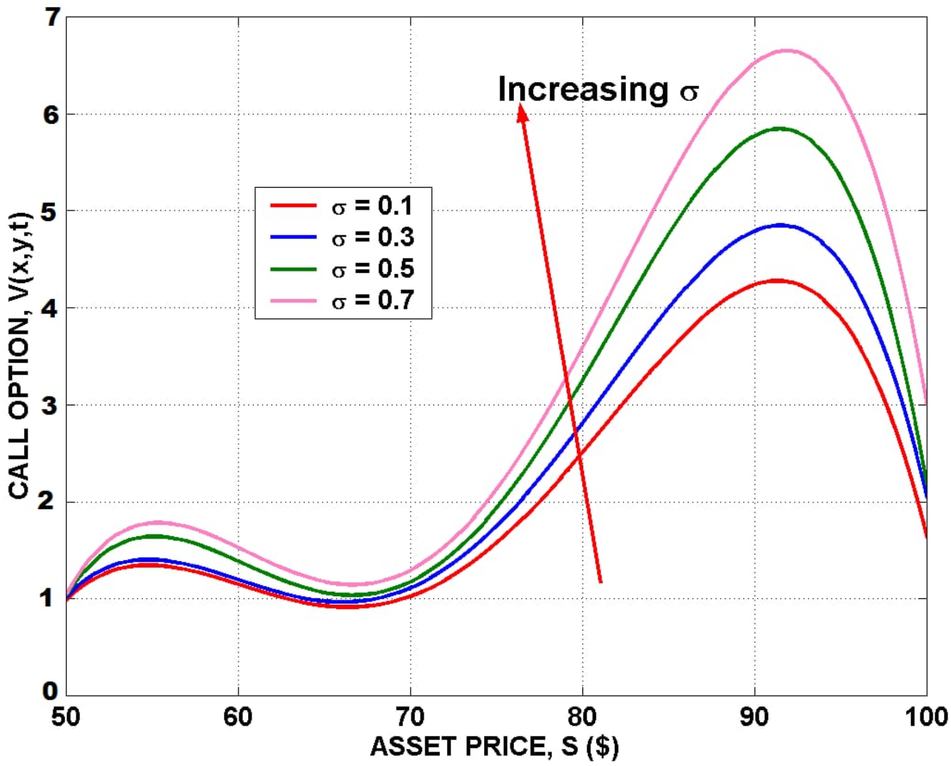

Figure 4.1 Graph of call option asset price and volatility

Figure 4.1 shows the variations of call option value (V) against the asset price (S). As the volatility increases ($\sigma = 0.1$, $\sigma = 0.2$, $\sigma = 0.3$ and $\sigma = 0.4$), the call option value (V) also increases at a constant underlying asset price (S). For instance at $S = 90$, V will be ($V = 4.3754$,$V = 4.9743$,$V = 5.92807$ and $V = 6.762397$) respectively. This is because an increase in volatility increases the likelihood that the price of the underlying asset will move in favour of the call option buyer. As a result the option becomes more valuable and its prices increases. This implies that the higher the volatility the more likelihood that the investors will assume that the call option value will rise. It also shows that there will be higher probability of higher price fluctuations.

In reality market uncertainties are also costs and therefore it will over value the call option. On the other hand, when volatility decreases, the price of the call option tends to decrease as well. This is because, a decrease in volatility decreases the likelihood that price of the underlying asset will move in favour of the call option buyer.

Consequently, the option becomes less valuable and its price decreases.

4.3 Effects of volatility on put option value

Equation (3.45) is solved using MATLAB and the results of the effects Volatility on call option value are presented as shown in table 4.2 below

Table 4.2
Put option values for varying Volatility

Volatility	Asset Price , S($)					
	50	60	70	80	90	100
$\sigma = 0.1$	16.7820634	42.256389	20.50747	11.9511605	9.011398	7.857236
$\sigma = 0.3$	18.054138	44.12857	21.205823	12.198779	9.126552	7.887723
$\sigma = 0.5$	19.32602	46.0019	21.904226	12.443050	9.19397	7.918147
$\sigma = 0.7$	20.5980	47.872902	22.60269	12.694126	9.985250	7.948697

The above results in Table 4.2 are presented in Figure 4.2 below

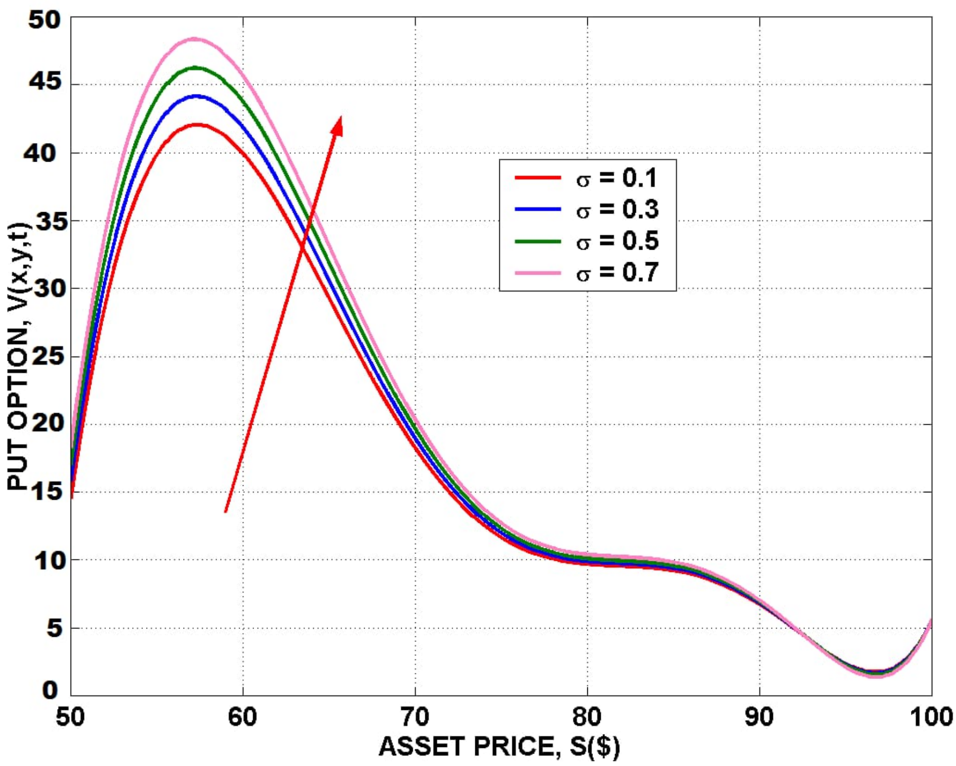

Figure 4.2 Graph of put option, asset price and volatility

Figure 4.2 shows the variations of put option value (V) against the asset price (S). As the volatility increases $(\sigma = 0.1, \sigma = 0.3, \sigma = 0.5$ and $\sigma = 0.7)$, the put option value (V) also increases at a constant underlying asset price (S). For instance at $S = 90, V$ will be $(V = 9.011398, V = 9.126552, V = 9.19397$ and $V = 9.985250)$ respectively. This means that an increase in volatility increases likelihood of larger price swings in the underlying asset. As a result there is a higher probability that price of the underlying asset will fall below strike price of put option which makes put option more valuable. This implies that the higher the volatility the more likelihood that the investors will assume that the put option value will rise. On the other hand, when volatility decreases, the likelihood of larger price swings decrease which makes the put option less valuable. It also shows that there will be higher probability of higher price fluctuations. In reality market uncertainties are also costs and therefore it will over value the put option.

4.4 Effects of Transaction Cost on Call Option Value

Equation (3.44) is solved using MATLAB and the results of the effects transaction cost on call option value are presented as shown in table 4.3 below.

Table 4.3
Call option values for varying transaction cost

Transaction Cost	Asset Price , S($)					
	50	60	70	80	90	100
$C = 5$	1.376309	1.4764298	1.9500810	3.1380863	5.359598	3.33276
$C = 10$	1.3495107	1.4272920	1.71495071	2.4380863	4.366086	2.30533
$C = 15$	1.337205	1.3947286	1.61160778	1.8632070	3.602757	1.758076
$C = 20$	1.324810	1.38676281	1.32183076	1.1471211	2.839362	1.6106954

The above results in Table 4.3 are presented in Figure 4.3

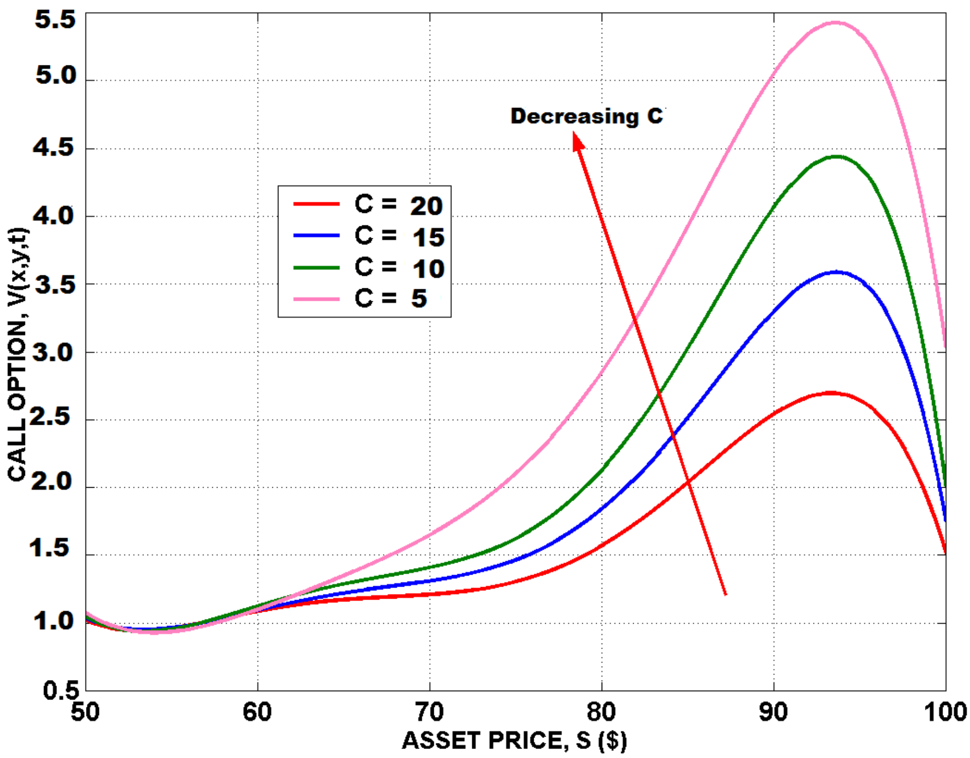

Figure 4.3 Graph of call option , asset price and transaction cost

In Figure 4.3 it is observed that as transaction cost increases $(C = 5, C = 10, C = 15$ and $C = 20)$ the call option value (V) decreases, for example, at $S = 90$, V will be; $(V = 5.359598, V = 4.366086, V = 3.602757$ and $V = 2.839362)$. The call option-asset price graph and transaction cost shows an inverse relation, that is, as transaction costs increase, call option values decrease. This graph is not a perfect linear relationship due to the varied impacts of transaction costs based on market conditions such as trading frequency and other related factors. Transaction cost can be described as fees and expenses incurred when trading. Examples of such costs are bid-ask spreads, market impact costs, brokerage commission among others. Brokerage commission as a transaction cost is the fees charged by brokers for executing trades.

An increase in commission in executing a trade may result into an increase in the cost of trading option consequently reducing the option value. Another factor that can bring about an increase in transaction cost is the bid-ask spread. Bid price is the highest price a buyer is willing to pay for the security while ask price is the price at which the sellers are willing to sell a particular security. The difference between the bid price and the ask price is the bid-ask spread. When there is a wider bid-ask spread, the cost of entering and exiting positions increase. This consequently reduces the profitability of option trades. The larger spreads implies that the traders have to pay a higher premium to buy options. This reduces option value. Market impact cost as an aspect of transaction cost can also reduce call option value. Market impact cost can be defined as additional cost that a trader must pay over the initial price due to market slippage. It is the cost incurred because the transaction itself changed the price of the asset. Larger market impact costs adversely affect price movements and increased transaction costs. Such costs consequently reduce call option values. The increase in transaction cost may discourage traders from buying an option. This could result into reduced liquidity and lower trading volumes in the option market. Increased transaction costs also reduce the possible profit that can be made from a call option. The higher the transaction cost the lower the potential profit and call option value.

Incorporation of transaction costs such as widening bid-ask spread, factoring in commission or taking into account market impacts, when valuing options can lead to significant changes in call option values. These aspects of transaction costs can affect the calculated option value and consequently trading strategies and risk management decisions. Investors and traders should be aware of transaction cost and consider their impact when evaluating option value.

4.5 Effects of Transaction Cost on put option value

Equation (3.45) is solved using MATLAB and get the results of the effects of transaction cost on put option value are presented as shown in table 4.4 below

Table 4.4
Put option values for varying transaction cost

Transaction Cost	Asset Price, S($)					
	50	60	70	80	90	100
$C = 5$	14.48008102	34.0498281	15.5760762	8.72862070	6.97235625	5.148657467
$C = 10$	12.14677105	26.0665286	12.342778	7.80036113	5.8696874	5.13705760
$C = 15$	9.81338107	19.032292	9.3093794	6.87202157	5.48905863	5.04536112
$C = 20$	7.48008109	12.0994298	7.6760810	5.64366981	5.11736954	5.11360035

The above results in Table 4.4 are presented in Figure 4.4

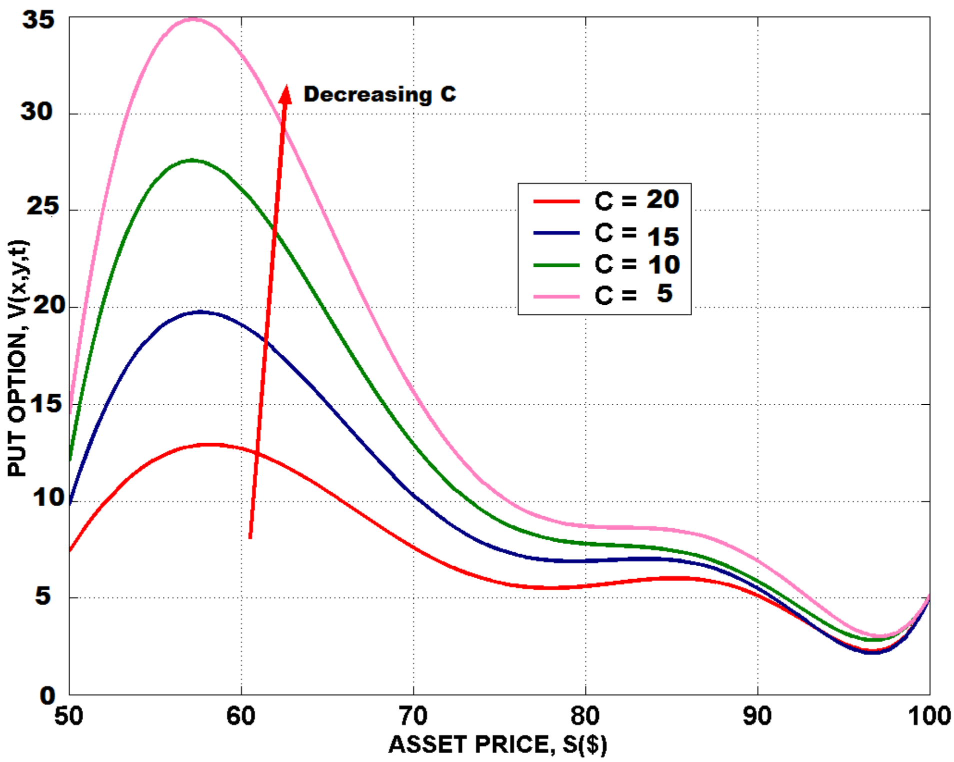

Figure 4.4 Graph of put option, asset price and transaction cost

In Figure 4.4 it is observed that as transaction cost increases ($C = 5$, $C = 10$, $C = 15$ and $C = 20$) the put option value (V) decreases, for example, at $S = 90$, V will be; ($V = 6.97235625$, $V = 5.8696874$, $V = 5.48905863$ and $V = 5.11736954$). The put option value-asset price graph shows a negative gradient with an inverse relationship between put option value, asset price and transaction cost. Transaction costs such as brokerage commission charged by brokers for executing trades may affect put option value. An increase in commission in executing a trade may result into an increase in the cost of trading option. The increase in the cost of trading eventually reduces possible profit and put option value. Another aspect which leads to high transaction cost is the trading frequency. High trading frequency brings about high transaction cost. This is attributed to increased trading activities. The multiple number of transactions reduction in potential put option value with time.

When transaction cost increases, the put option value decreases, this comes as a result of widening bid-ask spreads. The widened bid-ask spread reduces the probability of profit Increased transaction cost discourage traders from selling put options leading to reduced liquidity and lower trading volumes in the options market. Higher transaction costs reduce potential profit that can be made from put option. It is therefore important to consider transaction cost when evaluating the profitability and feasibility of options.

4.6 Validity and Reliability of the Results

This study applied Dufort-Frankel Numerical Method to approximate the solution to the Two-Dimensional Black-Scholes-Merton Partial Differential Equation with transaction cost and non-constant volatility. The findings from this study demonstrated that option values increase with an increase in volatility. The results positively identify with the findings in Liu et. al. (2019) on Finite Element Numerical Method and the Dimension Reduction technique in obtaining the approximate solution to Classical European option price under stochastic volatility which established that option values increase with an increase in volatility. Further, the present study established that an increase in transaction cost results to a decrease in the values of both call and put options. This finding agrees with the finding in the study by Sultan et. al. (2021) on effect of transaction costs on profit and capital formation of Soybean farming in Lamongan Regency which demonstrated that there is a negative correlation between transaction cost and profit.

The reliability of this study is demonstrated in addressing the defined objectives of this study. First, the study modified the original Black-Scholes equation to a two-dimensional Black-Scholes-Merton Partial Differential Equation with transaction cost and non-constant volatility.

Second, the study determined the effect of volatility on option value of the asset price, that is, as volatility increases, option values also increase. Similarly, as volatility decreases, option values decrease.

Lastly, the study established that as transaction cost increases, option values decrease. On the other hand, as transaction cost decreases, option values increase. This inverse correlation between transaction cost and option values is in line with the real trading situation in financial markets.

CHAPTER FIVE

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

5.1 Introduction

In this chapter, conclusions made on the basis of the results obtained and recommendations are presented. Further, recommendations for further research are suggested.

5.2 Summary

This section shows a summary of the achieved specific objectives of the study.

5.2.1 Mathematical Model of Black-Scholes-Merton partial differential with transaction cost and volatility

In this study, a mathematical model of Black-Scholes-Merton partial differential equation was developed using Dufort-Frankel numerical scheme as shown in equation (3.35) and indicated in this chapter as equation (5.1)

$$\frac{V_{i,j}^{n+1}-V_{i,j}^{n-1}}{2\Delta t}+(1-C)\left[\frac{V_{i+1,j}^{n}-V_{i-1,j}^{n}}{2\Delta x}+\frac{V_{i,j+1}^{n}-V_{i,j-1}^{n}}{2\Delta y}\right]+$$
$$\frac{\sigma}{2}\left[\frac{V_{i+j,j}^{n}-\left(V_{i,j}^{n+1}+V_{i,j}^{n-1}\right)+V_{i-j}^{n}}{(\Delta x)^2}\right]+\frac{\sigma}{2}\left[\frac{V_{i,+1}^{n}-\left(V_{i,j}^{n-1}+V_{i,j}^{n-1}\right)+V_{i,j-1}^{n}}{(\Delta y)^2}\right]$$

$$+\sigma^2\left[\frac{V_{i+1,+1}^{n}-2V_{i-1,+1}^{n}-V_{i+1j+1}^{n}+V_{i-1,-1}^{n}}{4\Delta x)(\Delta y)}\right]-V_{i,j}^{n}=0 \tag{5.1}$$

5.2.2 Effects of varying volatility on call and put option

As volatility increases, the call option value also increases at a constant underlying asset price whereas as volatility decreases, the value of call option also decreases. Similarly, put option value also increases with an increase on volatility and decreases with a decrease in volatility.

5.2.3 Effects of incorporating transaction cost on call and put option.

As transaction cost increases, the call option value of the underlying asset decreases. On the other hand, as transaction cost decreases, the value of call option increases. Similarly, an increase in the transaction cost also results into decrease in the value of put option value while a decrease in transaction cost results into an increase in put option value.

5.3 Conclusions

In this study, a mathematical model of Black-Scholes-Merton partial differential equation was developed using Dufort-Frankel numerical scheme and tested in MATLAB using arbitrary secondary data. The results from this study led to the conclusion that as volatility of the asset price increases, both the call and put option values of the underlying asset increase. On the other hand, as volatility of the asset price decreases, both the call and put option values of asset price decrease. This study further determined the effect of transaction cost on call and put option values of assets. The study demonstrated that an increase on transaction cost leads to a decrease on the value of both call and put options. On the other hand, a decrease on transaction cost leads to an increase on both call and put option values of the underlying asset. Expenses incurred while executing trade such as transaction cost can cause discrepancies between theoretical and actual option prices. It is therefore essential for traders to consider transaction costs when analyzing and trading options to ensure that they are making informed decisions.

Option pricing models such as Black-Scholes model can therefore be modified to include transaction costs and non-constant volatility by incorporating these parameters as additional variables so as to achieve a more accurate asset pricing.

5.4 Recommendations

This study developed a modified mathematical model of Black-Scholes-Merton partial differential equation with transaction cost and non-constant volatility.

The study recommended that it is imperative to make adjustments in the original Black-Schole model by relaxing the assumptions used so as to come up with a model which is reliable and more accurate in option valuation.

An increase on volatility of the asset price resulted into an increase on both call and put option values of asset while a decrease on volatility resulted into a decrease on both the call and put option values of the underlying asset. This study recommended that traders should carefully assess the market conditions and employ robust risk management strategies and adapt their trading approaches to navigate the challenges and opportunities presented by the volatile markets.

Based on the effect of transaction cost on option values, the model has demonstrated that an increase on transaction cost results into a decrease on option values while a decrease on transaction cost leads to an increase on option values. This study therefore recommended that the traders should avoid high trading frequency activities so as to minimise on the cost of executing trade and maximize possible return.

5.5 Suggestions for Further Studies

This study mainly focused on the numerical investigating the effects transaction cost and non-constant volatility on option values using a two-dimensional Black-Scholes-Merton Partial differential Equation, a case of European option. From the study the following are some suggested areas for further research:

(i) Investigating effects of transaction cost and non-constant volatility on American option pricing using Black- Scholes equation.

(ii) Proving the stability of Dufort and Frankel difference scheme in Black-Scholes equation

(iii) Comparison of option pricing values between the use of other valuation models and the Black-Scholes model

(iv) Investigate the option valuation a case of three dimensional nonlinear Black-Scholes equation.

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APPENDICES

Appendix I: Computer codes for option values

Computer codes for effects of transaction cost on put option values

```
» C=5
C =
    5
» A=[1/3 (C+1)/6 0 0 0 0;(C+1)/6 1/3 (C+1)/6 0 0 0;0 (C+1)/6 1/3 (C+1)/6 0 0;0 0
(C+1)/6 1/3 (C+1)/6 0;0 0 0 (C+1)/6 1/3 (C+1)/6;0 0 0 0 (C+1)/6 1/3
» U=[0.37;0.14;0.049;0.018;0.0067;0.0025]; » A*U
ans =
    14.48008102
    34.0498281
    15.5760762
    8.72862070
    6.97235625
    5.148657467

» C=10
C =
10
» A=[1/3 (C+1)/6 0 0 0 0;(C+1)/6 1/3 (C+1)/6 0 0 0;0 (C+1)/6 1/3 (C+1)/6 0 0;0 0
(C+1)/6 1/3 (C+1)/6 0;0 0 0 (C+1)/6 1/3 (C+1)/6;0 0 0 0 (C+1)/6 1/3];
» U=[0.37;0.14;0.049;0.018;0.0067;0.0025]; » A*U
```

ans =

12.14677105

26.0665286

12.342778

7.80036113

5.8696874

5.13705760

» C=15

C =

15

» A=[1/3 (C+1)/6 0 0 0 0;(C+1)/6 1/3 (C+1)/6 0 0 0;0 (C+1)/6 1/3 (C+1)/6 0 0;0 0 (C+1)/6 1/3 (C+1)/6 0;0 0 0 (C+1)/6 1/3 (C+1)/6;0 0 0 0 (C+1)/6 1/3];

» U=[0.37;0.14;0.049;0.018;0.0067;0.0025]; » A*U

ans =

9.81338107

19.032292

9.3093794

6.87202157

5.48905863

5.04536112

» C=20

C =

20

» A=[1/3 (C+1)/6 0 0 0 0;(C+1)/6 1/3 (C+1)/6 0 0 0;0 (C+1)/6 1/3 (C+1)/6 0 0;0 0 (C+1)/6 1/3 (C+1)/6 0;0 0 0 (C+1)/6 1/3 (C+1)/6;0 0 0 0 (C+1)/6 1/3];

» U=[0.37;0.14;0.049;0.018;0.0067;0.0025]; » A*U

ans =

7.48008109

12.0994298

7.6760810

5.64366981

5.11736954

5.11360035

Computer codes for effects transaction cost on call option values

» C=5

C =

5

» A=[1/3 (C+1)/6 0 0 0 0;(C+1)/6 1/3 (C+1)/6 0 0 0;0 (C+1)/6 1/3 (C+1)/6 0 0;0 0 (C+1)/6 1/3 (C+1)/6 0;0 0 0 (C+1)/6 1/3 (C+1)/6;0 0 0 0 (C+1)/6 1/3];

» U=[2.72;7.39;20.09;54.59;148.41;403.43]; » A*U

ans =

1.0e+003 *

1.376309

1.4764298

1.9500810

3.1380863

5.359598

3.33276

» C=10

C =

10

» A=[1/3 (C+1)/6 0 0 0 0;(C+1)/6 1/3 (C+1)/6 0 0 0;0 (C+1)/6 1/3 (C+1)/6 0 0;0 0 (C+1)/6 1/3 (C+1)/6 0;0 0 0 (C+1)/6 1/3 (C+1)/6;0 0 0 0 (C+1)/6 1/3];

» U=[2.72;7.39;20.09;54.59;148.41;403.43]; » A*U

ans =

1.0e+004 *

1.3495107

1.4272920

1.71495071

2.4380863

4.366086

2.30533

» C=15

C =

15

» A=[1/3 (C+1)/6 0 0 0 0;(C+1)/6 1/3 (C+1)/6 0 0 0;0 (C+1)/6 1/3 (C+1)/6 0 0;0 0 (C+1)/6 1/3 (C+1)/6 0;0 0 0 (C+1)/6 1/3 (C+1)/6;0 0 0 0 (C+1)/6 1/3];

» U=[2.72;7.39;20.09;54.59;148.41;403.43]; » A*U

ans =

1.0e+004 *

1.337205

1.3947286

1.61160778

1.8632070

3.602757

1.758076

» C=20

C =

20

» A=[1/3 (C+1)/6 0 0 0 0;(C+1)/6 1/3 (C+1)/6 0 0 0;0 (C+1)/6 1/3 (C+1)/6 0 0;0 0 (C+1)/6 1/3 (C+1)/6 0;0 0 0 (C+1)/6 1/3 (C+1)/6;0 0 0 0 (C+1)/6 1/3];

» U=[2.72;7.39;20.09;54.59;148.41;403.43];

» A*U

ans =

1.0e+004 *

1.324810

1.38676281

1.32183076

1.1471211

2.839362

1.6106954

Codes for the effects of volatility on call option values

» A=[1/3 (C+1)/6 0 0 0 0;(C+1)/6 1/3 (C+1)/6 0 0 0;0 (C+1)/6 1/3 (C+1)/6 0 0;0 0 (C+1)/6 1/3 (C+1)/6 0;0 0 0 (C+1)/6 1/3 (C+1)/6;0 0 0 0 (C+1)/6 1/3];

» y=0.1

y =

0.1000

» A=[1/(5-10*y) (99+10*y)/(10-20*y) 0 0 0 0;(99+10*y)/(10-20*y) 1/(5-10*y) (99+10*y)/(10-20*y) 0 0 0;0 (99+10*y)/(10-20*y) 1/(5-10*y) (99+10*y)/(10-20*y) 0 0;0 0 (99+10*y)/(10-20*y) 1/(5-10*y) (99+10*y)/(10-20*y) 0;0 0 0 (99+10*y)/(10-20*y) 1/(5-10*y) (99+10*y)/(10-20*y); 0 0 0 0 (99+10*y)/(10-20*y) 1/(5-10*y)]; » U=[2.72;7.39;20.09;54.59;148.41;403.43];

» A*U

ans =

1.0e+003 *

1.0190

1.1285

1.2910

2.2424 4.3754

1.5989

» y=0.3

y =

0.3000

» A=[1/(5-10*y) (99+10*y)/(10-20*y) 0 0 0 0;(99+10*y)/(10-20*y) 1/(5-10*y) (99+10*y)/(10-20*y) 0 0 0;0 (99+10*y)/(10-20*y) 1/(5-10*y) (99+10*y)/(10-20*y) 0 0;0 0 (99+10*y)/(10-20*y) 1/(5-10*y) (99+10*y)/(10-20*y) 0;0 0 0 (99+10*y)/(10-20*y) 1/(5-10*y) (99+10*y)/(10-20*y); 0 0 0 0 (99+10*y)/(10-20*y) 1/(5-10*y)]; » A*U

ans =

1.0e+003 *

1.0615

1.1607

1.4759

2.4569 4.9743

1.9975

» y=0.5

y =

0.5000

» A=[1/(5-10*y) (99+10*y)/(10-20*y) 0 0 0 0;(99+10*y)/(10-20*y) 1/(5-10*y) (99+10*y)/(10-20*y) 0 0 0;0 (99+10*y)/(10-20*y) 1/(5-10*y) (99+10*y)/(10-20*y) 0 0;0 0 (99+10*y)/(10-20*y) 1/(5-10*y) (99+10*y)/(10-20*y) 0;0 0 0 (99+10*y)/(10-20*y) 1/(5-10*y) (99+10*y)/(10-20*y); 0 0 0 0 (99+10*y)/(10-20*y) 1/(5-10*y)];

```
» U=[2.72;7.39;20.09;54.59;148.41;403.43];
» A*U
» A=[1/5-10*y 99+10*y/10-20*y 0 0 0 0;99+10*y/10-20*y 1/5-10*y 99+10*y/10-20*y 0 0 0;0 99+10*y/10-20*y 1/5-10*y 99+10*y/10-20*y 0 0;0 0 99+10*y/10-20*y 1/5-10*y 99+10*y/10-20*y 0;0 0 0 99+10*y/10-20*y 1/5-10*y 99+10*y/10-20*y; 0 0 0 0 99+10*y/10-20*y 1/5-10*y]; » U=[2.72;7.39;20.09;54.59;148.41;403.43];
» A*U
ans =
    1.0e+004 *
    1.0678
    1.2409
    1.5553
    2.7819
    5.92807
    2.13460
» y=0.7
y =
    0.7000
» A=[1/5-10*y 99+10*y/10-20*y 0 0 0 0;99+10*y/10-20*y 1/5-10*y 99+10*y/10-20*y 0 0 0;0 99+10*y/10-20*y 1/5-10*y 99+10*y/10-20*y 0 0;0 0 99+10*y/10-20*y 1/5-10*y 99+10*y/10-20*y 0;0 0 0 99+10*y/10-20*y 1/5-10*y 99+10*y/10-20*y; 0 0 0 0 99+10*y/10-20*y 1/5-10*y]; » U=[2.72;7.39;20.09;54.59;148.41;403.43];
» A*U
ans =
    1.0e+004 *
    1.0931
    1.28769
```

1.779731

3.119986

6.762397

2.953109

Codes for the effects of volatility on put option values

» y=0.1

y =

0.1000

» A=[1/5-10*y 99+10*y/10-20*y 0 0 0 0;99+10*y/10-20*y 1/5-10*y 99+10*y/10-20*y 0 0 0;0 99+10*y/10-20*y 1/5-10*y 99+10*y/10-20*y 0 0;0 0 99+10*y/10-20*y 1/5-10*y 99+10*y/10-20*y 0;0 0 0 99+10*y/10-20*y 1/5-10*y 99+10*y/10-20*y; 0 0 0 0 99+10*y/10-20*y 1/5-10*y]; » U=[0.37;0.14;0.049;0.018;0.0067;0.0025];

» A*U

ans =

16.7820634

42.256389

20.50747

11.9511605

9.011398

7.857236

» y=0.3

y =

0.3000

» A=[1/5-10*y 99+10*y/10-20*y 0 0 0 0;99+10*y/10-20*y 1/5-10*y 99+10*y/10-20*y 0 0 0;0 99+10*y/10-20*y 1/5-10*y 99+10*y/10-20*y 0 0;0 0 99+10*y/10-20*y 1/5-10*y 99+10*y/10-20*y 0;0 0 0 99+10*y/10-20*y 1/5-10*y 99+10*y/10-20*y; 0 0 0 0 99+10*y/10-20*y 1/5-10*y]; » U=[0.37;0.14;0.049;0.018;0.0067;0.0025]; » A*U

ans =

18.054138

44.12857

21.205823

12.198779

9.126552

7.887723

» y=0.5

y =

0.5000

» A=[1/5-10*y 99+10*y/10-20*y 0 0 0 0;99+10*y/10-20*y 1/5-10*y 99+10*y/10-20*y 0 0 0;0 99+10*y/10-20*y 1/5-10*y 99+10*y/10-20*y 0 0;0 0 99+10*y/10-20*y 1/5-10*y 99+10*y/10-20*y 0;0 0 0 99+10*y/10-20*y 1/5-10*y 99+10*y/10-20*y; 0 0 0 0 99+10*y/10-20*y 1/5-10*y]; » U=[0.37;0.14;0.049;0.018;0.0067;0.0025]; » A*U

ans =

19.32602

46.0019

21.904226

12.443050

9.19397

7.918147

» y=0.7

y =

0.7000

» A=[1/5-10*y 99+10*y/10-20*y 0 0 0 0;99+10*y/10-20*y 1/5-10*y 99+10*y/10-20*y 0 0 0;0 99+10*y/10-20*y 1/5-10*y 99+10*y/10-20*y 0 0;0 0 99+10*y/10-20*y

```
1/5-10*y 99+10*y/10-20*y 0;0 0 0 99+10*y/10-20*y 1/5-10*y 99+10*y/10-20*y; 0 0 0
0 99+10*y/10-20*y 1/5-10*y]; » U=[0.37;0.14;0.049;0.018;0.0067;0.0025];
   » A*U
   ans =
      20.5980
      47.872902
      22.60269
      12.694126
      9.985250
      7.948697
```

Appendix II: Computer codes for graphs

Codes for graph of effects of transaction cost on put option

```
x=[50 60 70 80 90 100 ];
y=[14.48 34.05 15.58 8.72 6.97 5.15];
p = polyfit(x,y,8)
x2 = 50:0.001:100;
y2 = polyval(p,x2);
plot(x,y,x2,y2)
hold on
y=[12.15 26.07 12.34 7.80 5.87 5.14];
p = polyfit(x,y,6)
x2 = 50:0.001:100;
y2 = polyval(p,x2);
plot(x,y,x2,y2)
hold on
y=[9.81 19.03 9.31 6.87 5.49 5.05];
```

```
p = polyfit(x,y,5)
x2 = 50:0.001:100;
y2 = polyval(p,x2);
plot(x,y,x2,y2)
hold on
y=[7.48 12.10 7.68 5.64 5.12 5.11];
p = polyfit(x,y,5)
x2 = 50:0.001:100;
y2 = polyval(p,x2);
plot(x,y,x2,y2)
```

Codes for the graph of effects of transaction cost on call option

```
x=[50 60 70 80 90 100 ];
y=[1.38 1.48 1.95 3.14 5.34 3.33];
p = polyfit(x,y,6)
x2 = 50:0.001:100;
y2 = polyval(p,x2);
plot(x,y,x2,y2)
hold on
y=[1.35 1.43 1.71 2.44 4.37 2.31];
p = polyfit(x,y,6)
x2 = 50:0.001:100;
y2 = polyval(p,x2);
plot(x,y,x2,y2)
hold on
y=[1.33 1.39 1.61 1.86 3.60 1.76];
p = polyfit(x,y,6)
x2 = 50:0.001:100;
```

```
y2 = polyval(p,x2);
plot(x,y,x2,y2)
hold on
y=[1.32 1.39 1.32 1.15 2.84 1.61];
p = polyfit(x,y,5)
x2 = 50:0.001:100;
y2 = polyval(p,x2);
plot(x,y,x2,y2)
```

Codes for the graph of effects of volatility on call option values

```
    x=[50 60 70 80 90 100 ];
y=[1.02 1.13 1.29 2.24 4.38 1.60];
p = polyfit(x,y,6)
x2 = 50:0.001:100;
y2 = polyval(p,x2);
plot(x,y,x2,y2)
hold on
y=[1.06 1.16 1.48 2.46 4.97 2.00];
p = polyfit(x,y,6)
x2 = 50:0.001:100;
y2 = polyval(p,x2);
plot(x,y,x2,y2)
hold on
y=[1.07 1.24 1.55 2.78 5.93 2.13];
p = polyfit(x,y,6)
x2 = 50:0.001:100;
y2 = polyval(p,x2);
```

```
plot(x,y,x2,y2)
```

hold on

```
y=[1.09 1.29 1.77 3.12 6.76 2.95];
p = polyfit(x,y,6)
x2 = 50:0.001:100;
y2 = polyval(p,x2);
plot(x,y,x2,y2)
```

Codes for the graph of effects of volatility on put option

```
x=[50 60 70 80 90 100 ];
y=[16.78 42.26 20.50 11.95 9.01 7.86];
p = polyfit(x,y,6)
x2 = 50:0.001:100;
y2 = polyval(p,x2);
plot(x,y,x2,y2)
```

hold on

```
y=[18.05 44.13 21.21 12.20 9.13 7.89];
p = polyfit(x,y,6)
x2 = 50:0.001:100;
y2 = polyval(p,x2);
plot(x,y,x2,y2)
```

hold on

```
y=[19.32 46.00 21.9 12.44 9.19 7.91];
p = polyfit(x,y,6)
x2 = 50:0.001:100;
y2 = polyval(p,x2);
plot(x,y,x2,y2)
```

hold on

```
y=[ 20.60 47.87 22.60 12.69 9.99 7.95];
p = polyfit(x,y,6)
x2 = 50:0.001:100;
y2 = polyval(p,x2);
plot(x,y,x2,y2)
```

Appendix III: UoK Research Permit

UNIVERSITY OF KABIANGA

ISO 9001:2015 CERTIFIED

OFFICE OF THE DIRECTOR, BOARD OF GRADUATE STUDIES

REF: PGC/AM/001/19

DATE: 15TH JUNE, 2022

Calvince Ochieng Fwaga,
MAPS,
University of Kabianga,
P.O Box 2030- 20200,
KERICHO.

Dear Mr. Fwaga,

RE: CLEARANCE TO COMMENCE FIELD WORK

I am glad to inform you that the Board of Graduate Studies during its meeting on 11th May 2022 approved your research proposal entitled **"Numerical Investigation of Option Pricing Using Black Scholes Merton Partial Differential Equation with Non Constant Volatility and Transaction Cost"**.

I am also acknowledging receipt of your corrected proposal via email and hard copies. You are now free to commence your field work on condition that you obtain a research permit from NACOSTI.

Please note that, you are expected to publish at least one (1) paper in a peer reviewed journal before final examination (oral defense) of your Masters thesis.

Thank you.

Yours Sincerely,

UNIVERSITY OF KABIANGA
15 JUN 2022
BOARD OF GRADUATE STUDIES

Prof. J. K. Kibett
DIRECTOR, BOARD OF GRADUATE STUDIES.

cc 1. Dean, SST
2. HOD, MAPS
3. Supervisors

Appendix IV: NACOSTI Research license

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National Commission for Science, Technology and Innovation(NACOSTI),
Off Waiyaki Way, Upper Kabete,
P. O. Box 30623 - 00100 Nairobi, KENYA
Telephone: 020 4007000, 0713788787, 0735404245
E-mail: dg@nacosti.go.ke
Website: www.nacosti.go.ke

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Mathematical Investigation of Option Pricing using Black-Scholes-Merton Partial Differential Equation with Transaction Cost

Calvince Fwaga [a*], Wilys O. Mukuna [a] and Levi Otanga Olwamba [a]

[a] *Department of Mathematics Actuarial and Physical Sciences, University of Kabianga, Kenya.*

Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

Over the years studies have been done on option pricing valuation. The world market economies have experienced tremendous asset price fluctuations since 1980s. For this reason, efforts have been directed towards developing reliable and more accurate option pricing models. Black-Scholes-Merton model has so far been proved to be the most powerful and significant tool for the valuation of an option. However, its assumption of zero transaction cost on asset pricing yields inaccurate option values. The study investigates the effects of transaction cost on call and put option of an asset price using a two-dimensional Black-Scholes-Merton partial differential equation. The Dufort-Frankel Finite Difference Method is used to approximate the solution to the BSM model equation describing the value of an option with boundary conditions. The simulation is done with

Corresponding author: E-mail: fwagacal91@gmail.com;

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the aid of MATLAB software program. The effects of incorporating transaction cost on the two assets prices on the value of an option using BSMPDE are determined. From the study, it is established that as transaction cost increases, the call and put option values decrease. The effects of incorporating transaction cost on the values of call and put option are shown in tabular form and graphically. These results are useful to the investors in computing possible returns on investment based on more accurate asset pricing and to the government on policy formulation in controlling prices in stock exchange market.

Keywords: Black-Scholes-Merton partial differential equation; option value; dufort-Frankel scheme; Transaction cost.

1 Introduction

Before the discovery of Black-Scholes model by Fischer Black, Myron Scholes and Robert Merton in 1973, there was no standard method of option pricing agreed upon by the option traders. Traders majorly relied on their intuition to price options. A breakthrough was later seen when Black-Scholes-Merton model became the most powerful and significant tool for the valuation of an option. Fluctuation in market prices of assets prompted rigorous mathematical and probabilistic concepts through the theory of stochastic process, also referred to as Wiener process [1]. The concepts solved the challenges in option valuation and gave new mathematical ideas that provided solutions to problems in finance and other fields. [2] developed a modified Black-Scholes-Merton model for Option Pricing. The model provided flexibilities for the markets. The study suggested that conformable Black-Scholes-Merton model may provide a way of valuing European call option compared to classical Black-Scholes model and fractional Black-Scholes model.

The study further showed that a more robust statistical procedure is required to improve the accuracy of option valuation. Additionally, the study suggested that an aspect of interest in asset pricing model to be considered is that of the transaction cost. [3] studied numerical solution of linear and non-linear Black-Scholes option pricing equations by means of semidiscretization technique. The study revealed that for a linear case, a fourth order discretization with respect to the underlying asset variables allows a better accurate approximation solution while for the non-linear case of interest modeling option pricing with transaction cost, semi discretization technique provides a competitive numerical solution. The study revealed that in practice, transaction costs arise when trading securities. The results of the study further demonstrated that although such transaction costs are generally small for institution investors, their influence lead to significant increase in the option price. [4] studied analytical solution of fractional Black-Scholes European option pricing equation by using Laplace transform. The study combined the form of Laplace and the homotopy perturbation method to obtain a quick and accurate solution to the fractional Black-Scholes equation with boundary condition for a European option pricing problem. The proposed scheme found the solution without any discretization or restrictive assumptions and free from round off errors thereby reducing numerical computation to a greater extent.[5] studied the Lie algebraic approach for determining pricing for trade account options. The research examined the options for the trade account using the Lie symmetry analysis. The study demonstrated that Lie symmetry technique can be used to analyze systemic problems in financial field although the method requires rigorous solutions to the numerous algebraic expressions. [6] carried out a study on the Walrasian-Samuelson Price Adjustment Model. In the study, an Ito method for modeling the changes of the market value of securities traded due to new information which affects the market asset supply and demand was introduced. It is formulated on the basis of; market supply, demand functions and the equilibrium price (Walrasian price) adjustment assumption, that the proportional price increase is driven by excess demand. The study established that if the supply and demand curves turned to be linear from the point of equilibrium, then the process changes to become logistic equation of Brownian motion with Wiener type of the random element. The study revealed that allowing transaction cost to be stochastic improves the accuracy of the option price prediction. This study forms a basis of our research on option pricing where the transaction cost is not zero. [7] carried out a study on simple formulas for pricing and hedging European options in the Finite Moment Log-stable model. The study showed that as stability parameters increase, both call and put option also increase. The study however assumed that volatility is constant and no transaction cost is incurred in trading. [8] applied a comparative analytical approach and numerical technique to determine the price of call option and put option of an underlying asset in the frontier markets so as to predict stock price. The study modified the Black-Scholes model so as to determine parameters such as strike price and expiration time. Machine learning approach was applied using Rapidminer software. The approach showed better results over classical Black-Scholes Option Pricing model. The study further considered numerical calculation of volatility and established that as the price of stocks goes up due to overpricing, volatility also increases at a high rate. The study did not however considered other parameters affecting option value such as transaction costs. [9] investigated numerical solutions of the non-linear Black-Scholes partial differential equation which often appears in financial markets for European option pricing in the presence of transaction costs. The study exploited the transformations for the computational purpose of a nonlinear Black-Scholes partial differential equation to modify as a nonlinear parabolic type of partial differential equation with initial and boundary conditions for both call and put options. The study derived several schemes using Finite Volume Method and Finite Difference Method. The study established that both methods provide numerical solutions which are closer to exact solutions.

The novelity of this study is to develop a mathematical model of Black-Scholes-Merton partial differential equation with transaction cost using Dufort-Frankel numerical scheme and show the effects of incorporating transaction cost on option value.

The rest of this paper is arranged in such a way that: section 2 presents the mathematical formulation of the problem, section 3 presents the results of the study and discussions, section 4 presents the conclusion and recommendation and section 5 presents suggestion for further studies.

2 Mathematical Formulation

Considering the two-dimensional Black-Scholes-Merton differential equation;

$$\frac{\partial V}{\partial t}+r\left[\frac{\partial V}{\partial S_1}S_1+\frac{\partial V}{\partial S_2}S_2\right]+\frac{1}{2}\left[\sigma_1^2S_1^2\frac{\partial^2 V}{\partial S_1^2}+\sigma_2^2S_2^2\frac{\partial^2 V}{\partial S_2^2}\right]+\rho\sigma_1\sigma_2S_1S_2\frac{\partial^2 V}{\partial S_1\partial S_2}-rV=0 \tag{1}$$

Assume the two assets S_1 and S_2 considers the transaction costs C_1 and C_2 meaning that at a time dt each of the assets incurs a transaction cost C_1S_1dt and C_2S_2dt respectively, V representing the volatility and r is the risk free rate. Therefore adding the transaction costs on each of the underlying assets from equation (1) becomes

$$\frac{\partial V}{\partial t}+r\left[\frac{\partial V}{\partial S_1}S_1+\frac{\partial V}{\partial S_2}S_2\right]-\left[C_1S_1\frac{\partial V}{\partial S_1}+C_2S_2\frac{\partial V}{\partial S_2}\right]+\frac{1}{2}\left[\sigma_1^2S_1^2\frac{\partial^2 V}{\partial S_1^2}+\sigma_2^2S_2^2\frac{\partial^2 V}{\partial S_2^2}\right]$$

$$+\rho\sigma_1\sigma_2S_1S_2\frac{\partial^2 V}{\partial S_1\partial S_2}-rV=0 \tag{2}$$

Thus, equation (2) represents the standard two-dimensional BSMPDE considering the transactional costs(C) , volatility (σ) and correlation between the two assets (ρ). Using the transformations of independent variables [1] for example transforming from S_1 , S_2:
$x=In\left(S_1\right)-\left(r-\frac{1}{2}\delta_1^2\right)t$ and $y=In\left(S_2\right)-\left(r-\frac{1}{2}\delta_2^2\right)t$

Therefore, it will be transformed to:

$$\frac{\partial x}{\partial S_1}=\frac{1}{S_1},\ \ \frac{\partial x}{\partial S_2}=0,\ \frac{\partial y}{\partial S_1}=0,$$
$$\frac{\partial y}{\partial S_2}=\frac{1}{S_2},\frac{\partial t}{\partial S_1}=0,\frac{\partial t}{\partial S_2}=0 \tag{3}$$

Transforming the PDEs using the chain rule and replacing equation (3) into it from the 2D BSMPDE to obtain;

$$\frac{\partial V}{\partial S_1}=\frac{\partial V}{\partial x}\frac{\partial x}{\partial S_1}+\frac{\partial V}{\partial y}\frac{\partial y}{\partial S_1}+\frac{\partial V}{\partial t}\frac{\partial t}{\partial S_1}=\frac{1}{S_1}\frac{\partial V}{\partial x} \tag{4}$$

$$\frac{\partial V}{\partial S_2} = \frac{\partial V}{\partial x}\frac{\partial x}{\partial S_2} + \frac{\partial V}{\partial y}\frac{\partial y}{\partial S_2} + \frac{\partial V}{\partial t}\frac{\partial t}{\partial S_2} = \frac{1}{S_2}\frac{\partial V}{\partial y} \tag{5}$$

$$\frac{\partial^2 V}{\partial S_1^2} = \frac{\partial}{\partial S_1}\left[\frac{\partial V}{\partial S_1}\right] = \frac{\partial}{\partial S_1}\left[\frac{1}{S_1}\frac{\partial V}{\partial x}\right] = \frac{1}{S_1}\frac{\partial}{\partial S_1}\left[\frac{\partial V}{\partial x}\right] = \frac{1}{S_1}\frac{\partial}{\partial x}\left[\frac{1}{S_1}\frac{\partial V}{\partial x}\right] = \frac{1}{S_1^2}\frac{\partial^2 V}{\partial x^2} \tag{6}$$

$$\frac{\partial^2 V}{\partial S_2^2} = \frac{\partial}{\partial S_2}\left[\frac{\partial V}{\partial S_2}\right] = \frac{\partial}{\partial S_2}\left[\frac{1}{S_2}\frac{\partial V}{\partial y}\right] = \frac{1}{S_2}\frac{\partial}{\partial S_2}\left[\frac{\partial V}{\partial y}\right] = \frac{1}{S_2}\frac{\partial}{\partial y}\left[\frac{1}{S_2}\frac{\partial V}{\partial y}\right] = \frac{1}{S_2^2}\frac{\partial^2 V}{\partial y^2} \tag{7}$$

$$\frac{\partial^2 V}{\partial S_1 \partial S_2} = \frac{\partial}{\partial S_1}\left[\frac{\partial V}{\partial S_2}\right] = \frac{\partial}{\partial S_1}\left[\frac{1}{S_2}\frac{\partial V}{\partial y}\right] = \frac{1}{S_2}\frac{\partial}{\partial S_1}\left[\frac{\partial V}{\partial y}\right] = \frac{1}{S_2}\frac{\partial}{\partial y}\left[\frac{\partial V}{\partial S_1}\right] = \frac{1}{S_2}\frac{\partial}{\partial y} = \frac{1}{S_1 S_2} \tag{8}$$

Substituting the derivatives above into equation (8) and assuming $C_1 = C_2 = C$ which leads to:

$$\frac{\partial V}{\partial t} + (r - C)\left[\frac{\partial V}{\partial x} + \frac{\partial V}{\partial y}\right] + \frac{1}{2}\left[\sigma_1^2\frac{\partial^2 V}{\partial x^2} + \sigma_2^2\frac{\partial^2 V}{\partial y^2}\right] + \rho\sigma_1\sigma_2\frac{\partial^2 V}{\partial x \partial y} - rV = 0 \tag{9}$$

2.1 Dufort and Frankel difference scheme

The Dufort-Frankel scheme, proposed by [10] makes use of various themes discussed in Richardson and MADE schemes. It came into existence in an effort to address the instability associated with the Richardson scheme. The scheme is explicit, unconditionally stable and second order accurate in both the spatial and temporal dimensions. The Dufort-Frankel scheme is conditionally consistent with the partial differential equation it solves. It is unconditionally stable and easier to parallelize on high performance computer systems. However, since the Dufort-Frankel scheme is a two-step method, calculating the first temporal vector after the initial boundary requires some other method. The Dufort-Frankel scheme makes use of a time derivative estimation similar to the Richardson scheme.

2.2 Discretization of Black- Scholes -Merton equation

The governing equations for this particular problem are nonlinear in nature and there exists no analytical method of solving them hence a suitable numerical method is used. Finite method was used in this case.

2.3 Dufort and Frankel numerical scheme

We discretize the Black-Scholes-Merton option pricing partial differential equation (9) and form a Dufort- Frankel numerical scheme which we eventually solve using the finite difference method. Equation (9) is discretized to study the effects of transaction cost C, for call and put option values. In the Dufort-Frankel numerical scheme, V_t, V_{xx} , V_{yy} and V_{xy} are replaced by the central finite approximations but the value of $V_{i,j}^n$ in V_{xx} and V_{yy} are replaced by $\left(V_{i,j}^{n+1} + V_{i,j}^{n-1}\right)$ difference approximation. When these approximations are substituted into equation (9), and let $\sigma_1 = \sigma_2 = \sigma$, we get

$$\begin{gathered}\frac{V_{i,j}^{n+1} - V_{i,j}^{n-1}}{2\Delta t} + (1 - C)\left[\frac{V_{i+1,j}^n - V_{i-1,j}^n}{2\Delta x} + \frac{V_{i,j+1}^n - V_{i,j-1}^n}{2\Delta y}\right] + \\ \frac{\sigma}{2}\left[\frac{V_{i+j,j}^n - \left(V_{i,j}^{n+1} + V_{i,j}^{n-1}\right) + V_{i-j}^n}{(\Delta x)^2}\right] + \frac{\sigma}{2}\left[\frac{V_{i,+1}^n - \left(V_{i,j}^{n-1} + V_{i,j}^{n-1}\right) + V_{i,j-1}^n}{(\Delta y)^2}\right] \\ + \sigma^2\left[\frac{V_{i+1,+1}^n - 2V_{i-1,+1}^n - V_{i+1j+1}^n + V_{i-1,-1}^n}{4\Delta x)(\Delta y)}\right] - V_{i,j}^n = 0\end{gathered} \tag{10}$$

Taking $\phi = \frac{\Delta t}{(\Delta x)^2} = \frac{\Delta t}{(\Delta y)^2}, \Delta x = \Delta y$ on a square mesh and multiplying by 4Δ t throughout equation (10) and re-arranging, we get the scheme;

$(2-4\sigma)V_{i,j}^{n+1} = 0.04V_{i,j}^{n} - (0.2 - 0.2C + 2\sigma)V_{i+1,j}^{n} - (0.2 - 0.2C + 2\sigma)V_{i-1,j}^{n} + (2+2\sigma)V_{i,j}^{n-1} - (0.2 - 0.2C + 2\sigma)V_{i,j+1}^{n} - (0.2 + 0.2C + 2\sigma)V_{i,j-1}^{n} - \sigma^2 V_{i+1,j+1}^{n} + 2\sigma^2 V_{i-1,j+1}^{n}$

$$+\sigma^2 V_{i+1,j-1}^{n} - \sigma^2 V_{i-1,j-1}^{n} \tag{11}$$

Taking , $\Delta x = \Delta y = 0.1$, $\sigma = 0.1$ $\rho = r = 1$ and $\Delta t = 0.01, \Rightarrow \phi = 1$ and multiply by 10 we get the Dufort-Frankel scheme

$$V_{i,j}^{n+1} = 0.25V_{i,j}^{n}\frac{(2-C)}{8}\left(V_{i+1,j}^{n} + V_{i-1,j}^{n} + V_{i,j+1}^{n} + V_{i,j-1}^{n}\right) + \frac{(1.1)}{(1-02)}V_{i,j}^{n-1} + 0.005$$

$$\left(V_{i+1,j-1}^{n} + 2V_{i-1,j+1}^{n} - V_{i+1,j+1}^{n} - V_{i-1,j-1}^{n}\right) \tag{12}$$

Taking $n = 2, i = 2.\ldots.6, j = 1$, i.e. $S1 = S2$ the above scheme in equation (12) can be written in matrix form as

We use the initial conditions below in (13)

$$\begin{bmatrix} V_{1,1}^{2} \\ V_{2,1}^{2} \\ V_{3,1}^{2} \\ V_{4,1}^{2} \\ V_{5,1}^{2} \\ V_{6,1}^{2} \end{bmatrix} = \begin{bmatrix} 0.25 & -\frac{(2-C)}{8} & 0 & 0 & 0 & 0 \\ -\frac{(2-C)}{8} & 0.25 & -\frac{(2-C)}{8} & 0 & 0 & 0 \\ 0 & -\frac{(2-C)}{8} & 0.25 & -\frac{(2-C)}{8} & 0 & 0 \\ 0 & 0 & -\frac{(2-C)}{8} & 0.25 & -\frac{(2-C)}{8} & 0 \\ 0 & 0 & 0 & -\frac{(2-C)}{8} & 0.25 & -\frac{(2-C)}{8} \\ 0 & 0 & 0 & 0 & -\frac{(2-C)}{8} & 0.25 \end{bmatrix} \begin{bmatrix} V_{1,1}^{1} \\ V_{2,1}^{1} \\ V_{3,1}^{1} \\ V_{4,1}^{1} \\ V_{5,1}^{1} \\ V_{6,1}^{1} \end{bmatrix} + \begin{bmatrix} -\frac{(2-C)}{8}\left(V_{1,2}^{1} + V_{1,0}^{1}\right) + 1.375V_{1,1}^{0} + 0.005\left(V_{2,0}^{1} + 2V_{0,2}^{1} - V_{2,2}^{1} - V_{1,0}^{1}\right) \\ -\frac{(2-C)}{8}\left(V_{2,2}^{1} + V_{2,0}^{1}\right) + 1.375V_{2,1}^{0} + -0.005\left(V_{3,0}^{1} + 2V_{1,2}^{1} - V_{3,2}^{1} - V_{2,0}^{1}\right) \\ -\frac{(2-C)}{8}\left(V_{3,2}^{1} + V_{3,0}^{1}\right) + 1.375V_{3,1}^{0} + -0.005\left(V_{4,0}^{1} + 2V_{2,2}^{1} - V_{4,2}^{1} - V_{3,0}^{1}\right) \\ -\frac{(2-C)}{8}\left(V_{4,2}^{1} + V_{4,0}^{1}\right) + 1.375V_{4,1}^{0} + -0.005\left(V_{5,0}^{1} + 2V_{3,2}^{1} - V_{5,2}^{1} - V_{4,0}^{1}\right) \\ -\frac{(2-C)}{8}\left(V_{5,2}^{1} + V_{5,0}^{1}\right) + 1.375V_{5,1}^{0} + -0.005\left(V_{6,0}^{1} + 2V_{4,2}^{1} - V_{6,2}^{1} - V_{5,0}^{1}\right) \end{bmatrix} \tag{13}$$

$$\mathrm{V(x,y,0) = 0, t = 0, x \geq y} \tag{14}$$

The boundary conditions for call asset option are

$$\mathrm{V(x,0,t) = V(x,2,t) = 0,\ V(x,1,1) = e^{x}, t > 0, x \geq y} \tag{15}$$

While the boundary conditions for put asset option are

$$\mathrm{V(x,0,t) = V(x,2,t) = 0,\ V(x,1,1) = e^{-x}, t > 0, x \geq y} \tag{16}$$

For call option we use the conditions in equation (15) so that the matrix equation (13) becomes

$$\begin{bmatrix} V_{1,1}^{2} \\ V_{2,1}^{2} \\ V_{3,1}^{2} \\ V_{4,1}^{2} \\ V_{5,1}^{2} \\ V_{6,1}^{2} \end{bmatrix} = \begin{bmatrix} 0.25 & -\frac{(2-C)}{8} & 0 & 0 & 0 & 0 \\ -\frac{(2-C)}{8} & 0.25 & -\frac{(2-C)}{8} & 0 & 0 & 0 \\ 0 & -\frac{(2-C)}{8} & 0.25 & -\frac{(2-C)}{8} & 0 & 0 \\ 0 & 0 & -\frac{(2-C)}{8} & 0.25 & -\frac{(2-C)}{8} & 0 \\ 0 & 0 & 0 & -\frac{(2-C)}{8} & 0.25 & -\frac{(2-C)}{8} \\ 0 & 0 & 0 & 0 & -\frac{(2-C)}{8} & 0.25 \end{bmatrix} \begin{bmatrix} 0.36789 \\ 0.13533 \\ 0.04978 \\ 0.018316 \\ 0.006738 \\ 0.002479 \end{bmatrix} \tag{17}$$

For put option we use the conditions in equation (16) so that the matrix equation (13) becomes

$$\begin{bmatrix} V_{1,1}^2 \\ V_{2,1}^2 \\ V_{3,1}^2 \\ V_{4,1}^2 \\ V_{5,1}^2 \\ V_{6,1}^2 \end{bmatrix} = \begin{bmatrix} 0.25 & -\frac{(2-C)}{8} & 0 & 0 & 0 & 0 \\ -\frac{(2-C)}{8} & 0.25 & -\frac{(2-C)}{8} & 0 & 0 & 0 \\ 0 & -\frac{(2-C)}{8} & 0.25 & -\frac{(2-C)}{8} & 0 & 0 \\ 0 & 0 & -\frac{(2-C)}{8} & 0.25 & -\frac{(2-C)}{8} & 0 \\ 0 & 0 & 0 & -\frac{(2-C)}{8} & 0.25 & -\frac{(2-C)}{8} \\ 0 & 0 & 0 & 0 & -\frac{(2-C)}{8} & 0.25 \end{bmatrix} \begin{bmatrix} 2.71828 \\ 7.38906 \\ 20.0855 \\ 54.59518 \\ 148.41316 \\ 403.42879 \end{bmatrix} \tag{18}$$

Solving the above matrix equation (17) and (18), we get the solutions for effects of transaction cost C for call and put option values to get results.

3 Results and Discussions

The simulation results given focus on the effects of transaction cost C, on call and put asset option values.

3.1 Effects of Transaction Cost on call option value

We solve equation (17) using MATLAB and get the results of the effects transaction cost on call option value as shown in table 1 below:

Table 1. Value of a call option fluctuation for varying Transaction Cost.

Table 1. Call option values at varying transaction cost

2* Transaction Cost	Asset Price, S ($)					
	50	**60**	**70**	**80**	**90**	**100**
C = 5	1.376309	1.4764298	1.9500810	3.1380863	5.359598	3.33276
C = 10	1.3495107	1.4272920	1.71495071	2.4380863	4.366086	2.30533
C = 15	1.337205	1.3947286	1.61160778	1.8632070	3.602757	1.758076
C = 20	1.324810	1.38676281	1.32183076	1.1471211	2.839362	1.6106954

The above results in table 1 are presented in Fig. 1.

In Fig. 1, it is observed that as transaction cost increases ($C = 5, C = 10, C = 15$ and $C = 20$) the call option value (V) decreases, for example, at S = 90, V will be; ($V = 5.359598, V = 4.366086, V = 3.602757$ and $V = 2.839362$). The call option-asset price graph and transaction cost shows an inverse relation, that is, as transaction costs increase, call option values decrease. This graph is not a perfect linear relationship due to the varied impacts of transaction costs based on market conditions such as trading frequency and other related factors. Transaction cost can be described as fees and expenses incurred when trading. Examples of such costs are bid-ask spreads, market impact costs, brokerage commission among others. Brokerage commission as a transaction cost is the fees charged by brokers for executing trades. An increase in commission in executing a trade may result into an increase in the cost of trading option consequently reducing the option value. Another factor that can bring about an increase in transaction cost is the bid-ask spread. Bid price is the highest price a buyer is willing to pay for the security while ask price is the price at which the sellers are willing to sell a particular security. The difference between the bid price and the ask price is the bid-ask spread. When there is a wider bid-ask spread, the cost of entering and exiting positions increase. This consequently reduces the profitability of option trades. The larger spreads implies that the traders have to pay a higher premium to buy options. This reduces option value. Market impact cost as an aspect of transaction cost can also reduce call option value. Market impact cost can be defined as additional cost that a trader must pay over the initial price due to market slippage.

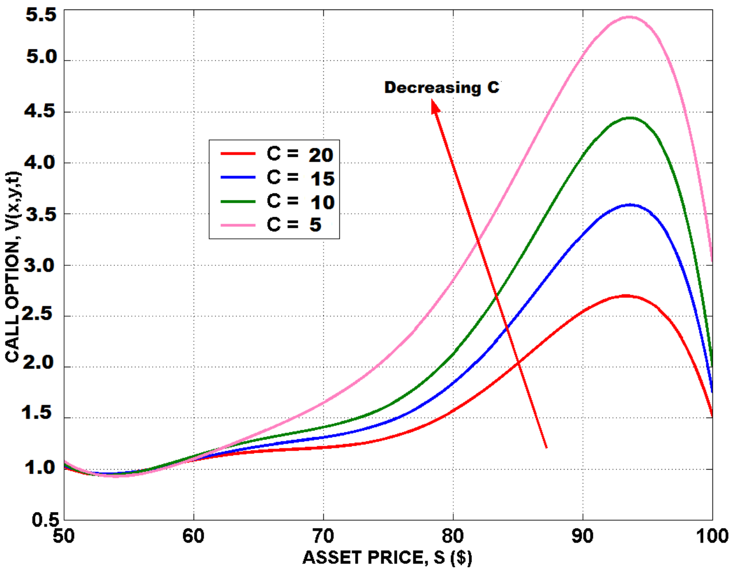

Fig. 1. Graph of call option value against asset price at varying transaction cost

It is the cost incurred because the transaction itself changed the price of the asset. Larger market impact costs adversely affect price movements and increased transaction costs. Such costs consequently reduce call option values. The increase in transaction cost may discourage traders from buying an option. This could result into reduced liquidity and lower trading volumes in the option market. Increased transaction costs also reduce the possible profit that can be made from a call option. The higher the transaction cost the lower the potential profit and call option value. Incorporation of transaction costs such as widening bid-ask spread, factoring in commission or taking into account market impacts, when valuing options can lead to significant changes in call option values. These aspects of transaction costs can affect the calculated option value and consequently trading strategies and risk management decisions. Investors and traders should be aware of transaction cost and consider their impact when evaluating option value.

3.2 Effects of Transaction Cost on put option value

We solve equation (18) using MATLAB and get the results of the effects transaction cost on call option value as shown in table 2 below

Table 2. Put option values at varying Transaction Cost

2* Transaction Cost	Asset Price, S ($)					
	50	**60**	**70**	**80**	**90**	**100**
C = 5	14.48008102	34.0498281	15.5760762	8.72862070	6.97235625	5.148657467
C = 10	12.14677105	26.0665286	12.342778	7.80036113	5.8696874	5.13705760
C = 15	9.81338107	19.032292	9.3093794	6.87202157	5.48905863	5.04536112
C = 20	7.48008109	12.0994298	7.6760810	5.64366981	5.11736954	5.01360035

The above results in table 2 are presented in Fig. 2.

In Fig. 2, it is observed that as transaction cost increases ($C = 5, C = 10, C = 15$ and C =20) the put option value (V) decreases, for example, at S=90, V will be; (V =6.97235625, V =5.8696874, V =5.48905863 and V =5.11736954). The put option value-asset price graph shows a negative gradient with an inverse relationship between put option value, asset price and transaction cost. Transaction costs such as brokerage commission charged by brokers for executing trades may affect put option value. An increase in commission in executing a trade may result into an increase in the cost of trading option. The increase in the cost of trading eventually reduces possible profit and put option value. Another aspect which leads to high transaction cost is the trading frequency. High trading frequency brings about high transaction cost. This is attributed to increased trading activities. The multiple number of transactions reduction in potential put option value with time. When transaction cost increases, the put option value decreases, this comes as a result of widening bid-ask spreads. The widened bid-ask spread reduces the probability of profit Increased transaction cost discourage traders from selling put options leading to reduced liquidity and lower trading volumes in the options market. Higher transaction costs reduce potential profit that can be made from put option. It is therefore important to consider transaction cost when evaluating the profitability and feasibility of options.

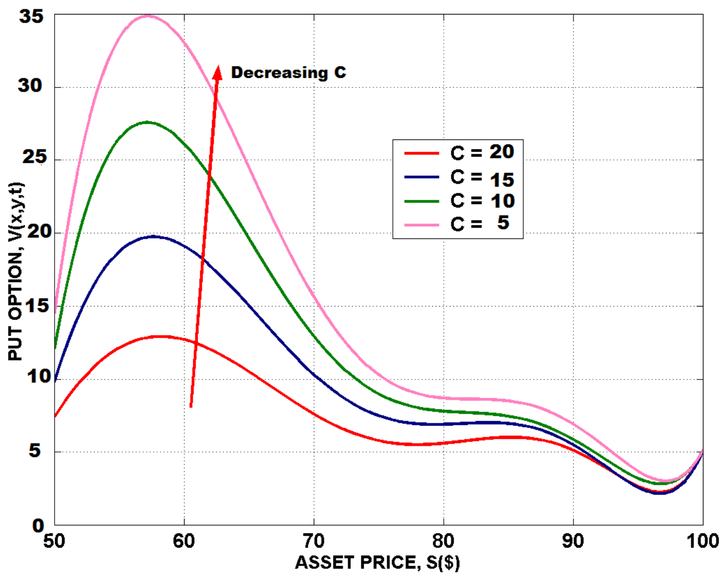

Fig. 2. Graph of put option value against asset price at varying Transaction Cost

4 Conclusion and Recommendation

4.1 Conclusion

The results from this study lead to the conclusion that incorporating the transaction cost on asset price results into a decrease on the value of both call and put option. Transaction cost can cause discrepancies between theoretical and actual option prices. It is therefore essential for traders to consider transaction costs when analyzing and trading options to ensure that they are making informed decisions. Option pricing models such as Black-Scholes model can therefore be modified to include transaction costs so as to achieve a more accurate asset pricing.

4.2 Recommendation

When pricing options, transactions costs need to be incorporated. This ensures a more accurate representation of real world trading scenarios. The findings from this study can be used by the potential investors to accurately compute possible returns on investment. To the government, the research findings can be used in formulating policies governing price control in stock exchange market.

5 Suggestion for Further Studies

The research recommends future work should consider a case of three-dimensional BSMPDE with transaction cost for American option.

Competing Interests

The authors declare that no competing interest regarding the publication of this paper.

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