LIE SYMMETRY SOLUTIONS OF SAWADA- KOTERA EQUATION

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DECLARATION AND APPROVAL

Declaration

This thesis is my original work and has not been submitted for the conferment of a degree or for the award of a diploma in this or any other university;-

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Approval

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DEDICATION

I dedicate this thesis with love, respect and gratitude to my dear husband Wesley, son Allan and daughters Annet and Annita for their support and encouragement.

ACKNOWLEDGEMENT

I thank God for His favour, mercies and guidance during the process of this work. I acknowledge with appreciation, the enthusiasm and professionalism of my supervisors Prof Maurice Oduor and Dr. John Rotich for their support, creative inspiration, mentorship skills as well as the profound competence without which it would have been extremely difficult for me to complete this work. I salute them for their tireless efforts, able leadership and great wealth of knowledge and experience.

ABSTRACT

The problems of differential equations are encountered in physical fields, engineering fields and mathematical world thus it is so important to find their exact solutions.The exact solutions of partial differential equations and ordinary differential equations have been sought by scholars for a number of decades. Researchers have used Lie symmetry approach to solve ordinary differential equations and partial differential equations. The progressive wave solution of one-dimensional wave equation was first discovered by Jeane Le Rond D' Atemmbert (1717-1783).His solution was a special application of the method of characteristics. The Sawada-Kotera equation is a special form of wave equation and the generalized Riccati equation mapping with the essential quotient group expansion techniques on constructing plentiful traveling wave results has been used in the past to solve the Sawada-Kotera equation among many other methods but the results the were not easily found since one could make errors during the plotting of graphs. In this study, we concentrated on analysis of fifth order Sawada-Kotera equation of the form; $u_t + 45u^2u_x + 15u_xu_{xx} + 15uu_{xxx} + u_{xxxxx} = 0$ using Lie symmetry analysis because the solution does not depend on the initial and boundary values hence is not an approximation to the exact solution and it has not been solved previously using this method. The study aimed at obtaining all the Lie groups admitted by the equation, invariant and exact solutions and symmetry solutions. The methodology involved application of infinitesimal transformations and generators, prolongations, adjoint symmetries, variation symmetries, invariant transformation and integrating factors so as to establish all the Lie groups shown by the equation.Our obtained solutions demonstrated that Lie symmetry analysis method is a sraight forward and best mathematical tool used to obtain analytical solutions of highly nonlinear PDEs.

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LIST OF NOTATIONS

P: Transformation group

α: Law of composition

m, n, k: Elements in the transformation group

m^{-1}: Inverse transformation element

e: Identity element

R: Set of all real numbers

A: An interval in the set of all real numbers

B: Domain within the set of all real numbers

X: A function

$\in, \delta$: Transformation parameters

C: Lie algebra vector field

D: Lie algebra set

x, t, u: Variables

T: Generator

g: Differential Equation

W: Solvable Lie algebra

α, β, λ: Functions of x, t and u

$\in$: Element Of

G: Generator

LIST OF ABBREVIATIONS AND ACRONYMS

HPM: Homotopy Perturbation Method

IVP: Initial Value Problem

KdV: Korteweg-de Vries equations

$MADM$: Modified Adomian Decomposition Method

$NLEE$: Nonlinear Evolution Equation

NLH: Nonlinear Helmholtz Equation

ODE: Ordinary Differential Equation

PDE: Partial Differential Equation

$q-HAM$: q-Homotopy Analysis Method

CHAPTER ONE

INTRODUCTION

1.1 Overview

This chapter entails the background of the study, basic concepts,statement of the problem, objectives of the study and the significance of the study.

1.2 Background of the Study

Lie symmetry group theory of differential equations has been in existence since 19th century. It was developed by Sophus Lie . He introduced the use of groups of transformations known as Lie groups in the research of differential equations' symmetry properties and their results. In this case a symmetry group outlines results of the system to another result of a similar system. Yaglom *et al.* [39].
Solving equations has been one of the most important driving forces in the history of Mathematics. Nonlinear PDEs and ODEs have been of great interest in the recent years because they are applied in physical, financial, engineering and mathematical fields since it is considered that solving problems of PDEs and ODEs is very important in applied Mathematics.
Lie did not only give the solution of the problems but also instituted a new branch of Mathematics in the field of symmetry.
In spite of the existence of literature on Lie groups, group theory has not been in use lately due to the following factors:
(i) Most scholars believe that there is difficulty in finding symmetry group of an equation as it is to solve it,
(ii) It is believed that Lie groups provide randomly occurring particular solutions only and

(iii) It is considered to be only useful for linear equations.

In this study, we have applied Lie symmetry analysis in the solution of Sawada-Kotera equation which is a fifth order nonlinear wave equation expressed as

$$u_t+45u^2u_x+15u_xu_{xx}+15uu_{xxx}+u_{xxxxx}=0 \tag{1.1}$$

1.3 Basic Mathematical Concepts

The following terms have been used in the study.

Definition 1.2.1 A Group

A group P is a non empty set of elements with a law of composition α between the elements of P which satisfy the conditions below [5]:

(i) Closure property

For elements m and n of $P, \alpha(m,n)$ is an element of P.

(ii) Associative property

For some elements m,n and k of $P, \alpha[m,\alpha(n,k)]=\alpha[\alpha(m,n),k]$

(iii) Identity element

There exist a unique identity element e of P such that for any element m of $P, \alpha(m,e)=\alpha(e,m)=m$

(iv) Inverse element

For some element m of P there exist a unique inverse element m^{-1} in P such that, $\alpha(m,m^{-1})=\alpha(m^{-1},m)=e$.

Definition 1.2.2 An Abelian Group

A group P is abelian if $\alpha(m,n)=\alpha(n,m)$ hold for all elements $m,n\in P$.[8]

Definition 1.2.3 A subgroup

A subgroup of P is a group formed by a subset of elements of P with the same law of composition α.

Definition 1.2.4 Point Transformation

Consider the point $X = (x_1, x_2, \ldots .x_n)$ which lies in the domain $B \subset R^n$ in n-dimensional space, then $x^* = f(x; \varepsilon)$ is a set of point transformation.

Definition 1.2.5 Group of Transformations

Let $X = (x_1, x_2 \ldots \ldots \ldots .x_n) \in B \subset R^n$. It follows that the set of transformation $x^* = f(x; \varepsilon)$ defined for every x in B depending upon a parameter ε lying in a set A $\subset R$ with $\alpha(\varepsilon, \delta)$ defining a law of composition of parameter ε and δ in A forms a set of transformations in B if and only if [6]:

(i) For every parameter $\varepsilon \in A$, the transformations are one to one and onto B and more specifically x^* lies in B.

(ii) A with the law of composition α forms a group P.

(iii) $X^* = X$ where $\varepsilon = e$ such that $X(x; e) = X$

(iv)If $x^* = f(x; \varepsilon), x^{**} = f(x^*; \delta)$ then $x^{**} = f(x; \alpha(\varepsilon, \delta))$.

Definition 1.2.6 An Orbit

An orbit of a point $X = (x_1, x_2, \ldots, x_n)$ is a set of points $x^* = x^*(\varepsilon)$ for every $\varepsilon \in A$

Definition 1.2.7 A One-parameter Lie Group

A group of transformation describes a one parameter Lie group of point transformation if besides satisfying the properties of $x^* = f(x;)$ as above, it also states that:

(i) A is an interval in R and ε is a continuous parameter; such that $\varepsilon = 0$ corresponding to the identity element e.

(ii) f is infinitely differentiable with respect to x in B and also a systematic function of ε in A.

(iii) $\alpha(\varepsilon, \delta)$ is a systematic function of $\varepsilon, \delta \in A$ and thus $\alpha(m, n) = m + n$ for lie groups.

Definition 1.2.8 A Two-parameter Lie Group

A two-parameter group of transformation $x^* = f(x; \varepsilon)$ with $x = (x_1, x_2 \ldots x_n)$ and parameters $\varepsilon = (\varepsilon_1, \varepsilon_2)$ is called a two- parameter lie group of transformation if it also satisfy the properties (i), (ii) above and the composition laws of parameters are denoted by $\phi(\varepsilon, \delta) = (\phi_1(\varepsilon, \delta), \phi_2(\varepsilon, \delta))$ which is an analytic function of $\varepsilon = (\varepsilon_1, \varepsilon_2)$ and $\delta = (\delta_1, \delta_2)$ in A.

Theorem 1.2.1 Lie's first fundamental theorem

There exists a parameterization $\beta(\varepsilon)$ such that the Lie group of transformations $x^* = f(x, \varepsilon)$ is equal to the solution of the initial value problem (IVP) for the first order differential equations

$\frac{dx^*}{d\beta} = \alpha(x^*)$ (1.2)

with initial conditions $x^* = x$, when $\beta = 0$ (1.3)

Particularly $\beta(\varepsilon) = \int_0^1 \alpha(\varepsilon')d\varepsilon'$ (1.4) where $\alpha(\varepsilon') = \frac{\partial\lambda(\varepsilon,\delta)}{\partial\delta}|_{(\varepsilon,\delta)=(\varepsilon^{-1},\varepsilon)}$ (1.5)

and $\alpha(0) = 1.$ (1.6)

whereby ε^{-1} denotes the inverse of ε.

Proof

First we show that $x^* = f(x, \varepsilon)$ leads to (1.2), (1.3), (1.4), (1.5). Expand the left hand side of $f(x; \varepsilon + \triangle\varepsilon) = f(f(x; \varepsilon); \lambda(\varepsilon^{-1}, \varepsilon + \triangle\varepsilon))$ (1.7)

in a power series in $\triangle\varepsilon$ about $\triangle\varepsilon = 0$ so that

$f(x; \varepsilon + \triangle\varepsilon) = x^* + \frac{\partial f(x;\varepsilon)}{\partial\varepsilon} \triangle \varepsilon + 0((\triangle\varepsilon)^2)$ (1.8)

where x^* is given by $x^* = f(x, \varepsilon)$. Then expanding $\lambda(\varepsilon^{-1}, \varepsilon+\triangle\varepsilon)$ in a power series in $\triangle\varepsilon$) about $\triangle\varepsilon = 0$ we have

$\lambda(\varepsilon^{-1}, \varepsilon + \triangle\varepsilon) = \lambda(\varepsilon^{-1}, \varepsilon) + \alpha(\varepsilon) \triangle \varepsilon + 0((\triangle\varepsilon)^2)$

$= \alpha(\varepsilon)\triangle\varepsilon{+}0((\triangle\varepsilon)^2)$ (1.9)

where $\alpha(\varepsilon)$ is defined by equation (1.5). Consequently, after expanding the right-hand side of equation (1.7) in a power series in $\triangle\varepsilon$ about $\triangle\varepsilon = 0$, we obtain

$$
\begin{aligned}
f(x; \varepsilon + \triangle\varepsilon) &= f(x^*; \lambda(\varepsilon^{-1}, \varepsilon + \triangle\varepsilon)) \\
&= f(x^*; \alpha(\varepsilon) \triangle \varepsilon 0((\triangle\varepsilon)^2)) \\
&= f(x^*; 0) + \alpha(\varepsilon) \triangle \varepsilon \frac{\partial f}{\partial\delta}(x^*; \delta)|_{\delta=0} + 0((\triangle\varepsilon)^2)) \\
&= x^* + \alpha(\varepsilon)\psi(x^*) \triangle \varepsilon + 0((\triangle\varepsilon)^2)). \qquad (1.10)
\end{aligned}
$$

Equating (1.8) and (1.9) we see that $x^* = f(x; \varepsilon)$ satisfies the initial value problem for the system of differential equations

$\frac{dx^*}{d\varepsilon} = \alpha(\varepsilon)\alpha(x^*)$ (1.11)

with $x^* = x$ at $\varepsilon = 0.$ (1.12)

From $x^* = x + \varepsilon\alpha(x) +^0 (\varepsilon^2)$ it follows that $\alpha(0) = 1$. The parameterization $\beta(\varepsilon) = \int_0^t \alpha(\varepsilon^{'})d\varepsilon^{'}$ leads to (1.2) and (1.3).

Since $\frac{\partial\alpha(x)}{\partial x_i}, i = 1, 2, 3...., n$ is continuous, it follows from the existence and uniqueness theorem for an (IVP) for a system of first order differential equations, that the solution of (1.2) and (1.3), and hence (1.11) and (1.12), exists and is unique.

This solution must be $x^* = f(x, \varepsilon)$, which completes the proof.

From the theorem, we assume that a one-parameter (ε) Lie group of transformations is parameterized such that its laws of composition $\lambda(\varepsilon, \delta) = \varepsilon + \delta$

And $\varepsilon^{-1} = \varepsilon$, where ε is the neutral element. That is the one-parameter Lie group of transformations $x^* = f(x, \varepsilon)$ now becomes;

$\frac{dx^*}{d\varepsilon} = \alpha(x^*)$

with initial conditions $x^* = x$, at $\varepsilon = 0$ (1.13)

where $\alpha(x)$ is the infinitesimal of $x^* = f(x, \varepsilon)$.

Definition 1.2.9 Vector Field

A vector field C on a set D allocate a tangent vector $C|x$ into every position $x \in D$, which are varying smoothly from position to position. In general coordinates $(x^1, \dots, x^m)$ a vector field is expressed in the form $C|x = \gamma^1(x)\partial/\partial x^1 + \gamma^2(x)/\partial x^2 + \dots + \gamma^m(x)\partial/\partial x^m$ in which γ^i is a smooth function of x which can be differentiable [27].

Definition 1.2.10 Commutator

If T_1 and T_2 are vector fields then their commutator also called Lie bracket is defined as $[T_1, T_2] = T_1T_2 - T_2T_1$

Example

Given two vector fields expressed as:

$T_1 = \frac{\partial}{\partial x}$

$T_2 = x\frac{\partial}{\partial x} + \frac{6}{7}y\frac{\partial}{\partial y}$

The commutator of these vector fields is

$[T_1, T_2] = (\frac{\partial}{\partial x})(x\frac{\partial}{\partial x} + \frac{6}{7}y\frac{\partial}{\partial y}) - (x\frac{\partial}{\partial x} + \frac{6}{7}y\frac{\partial}{\partial y})(\frac{\partial}{\partial x})$

$= (\frac{\partial(x)}{\partial x})\frac{\partial}{\partial x} + x\frac{\partial^2}{\partial x^2} + \frac{6}{7}\frac{\partial(y)}{\partial x})\frac{\partial}{\partial y} + \frac{6}{7}y\frac{\partial^2}{\partial x\partial y} - x\frac{\partial^2}{\partial x^2} - \frac{6}{7}y\frac{\partial^2}{\partial x\partial y}$

$= \frac{\partial}{\partial x} + x\frac{\partial^2}{\partial x^2} + 0 + \frac{6}{7}y\frac{\partial^2}{\partial x\partial y} - x\frac{\partial^2}{\partial x^2} - \frac{6}{7}y\frac{\partial^2}{\partial x\partial y}$

$= \frac{\partial}{\partial x}$

$= T_1$

Definition 1.2.11 Lie Algebra

A Lie algebra W is a vector space, on which commutation is defined and satisfies the following properties:

(i) Closure $T_1, T_2 \in W$ such that $[T_1, T_2] \in W$

(ii) Antisymmetry $[T_1, T_2] = -[T_2, T_1]$

(iii) Bilinearity $[k_1T_1 + k_2T_2, T_3] = k_1[T_1, T_2] + k_2[T_2, T_3]$ and $[T_1, k_1T_2 + k_2T_3] = k_1[T_1, T_2] + k_2[T_1, T_3]$ where k_1 and k_2 are constants.

(iv) Jacobi identity $[T_1, [T_2, T_3]] + [T_2, [T_3, T_1]] + [T_3, [T_1, T_2]] = 0$ for all T_1, T_2 and T_3 in W.

If $[T_1, T_2] = 0$ then T_1 and T_2 commute and when all elements of W commute, W is known as an abelian lie algebra.

Definition 1.2.12 Solvable Lie Algebra

A solvable lie algebra W with the series that are derived as

$W \supset W' = [W, W]$

$\supset W'' = [W', W']$

$\supset W''' = [W'', W'']$

$\supset \ldots.$

$\supset W^{(a)} = [W^{(a-1)}, W^{(a-1)}]$

such that $W^{(a)} = 0$ for some $a > 0$.

Theorem 1.2.2 Lie's Second Fundamental Theorem

The commutator of some two given infinitesimal generators of a k-parameter Lie group of transformations is also an infinitesimal generator, in particular

$$[w_i, w_j] = c_{ij}^n w_n \tag{1.14}$$

Whereby the coefficients c_{ij}^n are constants and $i, j, n = 1, 2, 3, \ldots..k$

For any given three infinitesimal generators w_i, w_j, w_n it is always true that

$$[w_i, [w_j, w_n]] + [w_j, [w_n, w_i]] + [w_n, [w_i, w_j]] = 0 \tag{1.15}$$

Equation (1.15) represents Jacobi's identity.

For proof, see [28].

Theorem 1.2.3 Lie's Third Fundamental Theorem

The constants, described by the commutation (1.2) satisfy the relations

$$c_{ij}^n = c_{ji}^n \tag{1.16a}$$

$$c_{ij}^k c_{kn}^m + c_{jn}^k c_{km}^m + c_{ni}^k c_{kj}^m = 0 \tag{1.16b}$$

and $[\alpha w_i + \beta w_j, w_l] = \alpha[w_i, w_j] + \beta[w_j, w_l], [w_i, \alpha w_j + \beta w_l] = \alpha[w_i, w_j] + \beta[w_i, w_l]$

For proof, see [6].

Infinitesimal generators (w_i) for $i = 1, 2, 3, ..., n$ described above, satisfy bilinear property in the commutator equations given as

$$[\alpha w_i + \beta w_j, w_l] = \alpha[w_i, w_j] + \beta[w_j, w_l],$$

$$[w_i, \alpha w_j + \beta w_l] = \alpha[w_i, w_j] + \beta[w_i, w_l] \tag{1.16c}$$

Definition 1.2.13 Infinitesimal Transformation

Consider a one parameter transformation $\overline{x} = X(x, y; \varepsilon)$ and $\overline{y} = Y(x, y; \varepsilon)$ in which ε is a continuous parameter. Expansion of this transformation using Taylor's series at the point $\varepsilon = \varepsilon_0$ yields

$$\overline{x} = x + (\tfrac{\partial X}{\partial \varepsilon})_{\varepsilon=\varepsilon_0}(\varepsilon - \varepsilon_0) + \ldots$$

$\overline{y} = y + (\frac{\partial Y}{\partial \varepsilon})_{\varepsilon=\varepsilon_0}(\varepsilon - \varepsilon_0) + \ldots$

Considering the group parameter ε that is evaluated at $\varepsilon = \varepsilon_0$,the partial derivatives are known to be infinitesimals and are functions of x and y. The study expresses them as.

$(\frac{\partial X}{\partial \varepsilon})_{\varepsilon=\varepsilon_0} = \alpha(x, y)$

$(\frac{\partial Y}{\partial \varepsilon})_{\varepsilon=\varepsilon_0} = \lambda(x, y)$

Considering the values of ε tending closer to ε_0, the coordinates of the transformation can be expressed as;

$\overline{x} = x + \mu(x, y)(\varepsilon - \varepsilon_0)$

$\overline{y} = y + \lambda(x, y)(\varepsilon - \varepsilon_0)$

such that the terms of second and higher degree in $(\varepsilon - \varepsilon_0)$ have been left out. Thus this transformation is known as an infinitesimal transformation [7,10].

Definition 1.2.14 Invariance under Transformation

An element or set of elements which does not change when its constituents change is called an invariant. Its concept has a physical basis in the conservation laws of mechanics. A function g is known to be invariant under a Lie group if and only if

$g(\overline{x}, \overline{y}) = g(X(x, y, \varepsilon), Y(x, y, \varepsilon)) = g(x, y)$

such that when expressed in new variables, the function reads the same.

Rotation of a circle about an axis that is normal to its center is a good example of invariance under a continuous transformation [7].

Definition 1.2.15 Symmetry

An operation which leaves invariant an object which it operates and a transformation which makes the object unchanged is called symmetry of a geometrical object. Considering the transformation of infinitesimal form

$\overline{x}_i = x_i + \varepsilon\alpha_i \; i = 1, \ldots, n$

where ε represents a parameter of smallness. This equation can be expressed as

$\overline{x}_i = (1 + \varepsilon T)x_i$

in which

$T = \alpha_i \frac{\partial}{\partial x_i}$

is a differential operator known as the generator of the transformation.

Regarding a particular case where

$T = \alpha \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial t} + \lambda \frac{\partial}{\partial u}$

Under the action of the infinitesimal transformation that is generated by T, a function $g(x, t, u)$ becomes

$\overline{g}(\overline{x}, \overline{t}, \overline{u}) = (1 + \varepsilon T)g(x, t, u)$

$= g + \varepsilon(\alpha \frac{\partial g}{\partial x} + \beta \frac{\partial g}{\partial t} + \lambda \frac{\partial g}{\partial u})$

If the form of g is unchanged such that

$\alpha \frac{\partial g}{\partial x} + \beta \frac{\partial g}{\partial t} + \lambda \frac{\partial g}{\partial u} = 0$

or

$\overline{g}(\overline{x}, \overline{t}, \overline{u}) = g(x, t, u)$

then T is known as a symmetry of g. In mathematical terms, all symmetries represent invariance under transformations. Examples of these symmetries may be reflections, translations and rotations which are referred generally to as geometric symmetries. Nonetheless, there are symmetries that may not have such a simple geometrical interpretation.

1.4 Statement of the Problem

The results of fifth order nonlinear wave equation can be analytic or numerical when finite difference approach is used whereby the convergence of numerical systems depend on initial and boundary values given.

Hasibun *et al.* [14] applied a generalized Riccati equation mapping with the essential (G'/G)-expansion technique on constructing abundant travelling wave results in a consistent manner for the fifth-order Sawada-Kotera equation but the results were not easily realized since errors could be made when plotting the graphs. In this study we have solved the fifth order Sawada-Kotera equation of the form

$$u_t + 45u^2u_x + 15u_xu_{xx} + 15uu_{xxx} + u_{xxxxx} = 0 \tag{1.1}$$

analytically using Lie symmetry analysis. The technique is among the most powerful approaches currently used to achieve precise solutions of the partial differential equations that are nonlinear and the solution is independent of either initial or boundary values hence is not an approximation to exact solution.

1.5 Objectives of the Study

1.5.1 General Objectives

The general objective was to find the general solution of the Sawada-Kotera equation, (1.1) using Lie symmetry analysis.

1.5.2 Specific Objectives

The specific objectives were to:

(i) Find the extensions of the generator and the total derivatives of the parameters in Sawada-Kotera equation.

(ii) Find the infinitesimal generators and the groups in which Sawada-Kotera equation admits.

(iii) Generate invariant and exact solutions of Sawada-Kotera equation.

(iv) Obtain symmetry solutions of Sawada-Kotera equation.

1.6 Significance of the Study

Nonlinear differential equations play a very significant function in the study of physical phenomena such as fluid flow and electromagnetics. Sawada-Kotera equation is widely used to model applications such as navigation, mechanical waves, sound waves, light waves and water waves.

The solutions of this study proof that Lie symmetry analysis is an alternative method of solving the fifth order nonlinear wave equation and attempts can be made to solve similar equations using this method. The study is the main contribution to knowledge and further research.

CHAPTER TWO

LITERATURE REVIEW

2.1 Introduction

This study presented some results of the previous related research done by some scholars.

Studies of PDEs and ODEs have been done by many scholars with the aim of finding their exact solutions. Since the problems of differential equations are encountered in both scientific, engineering and mathematical world, getting their exact solutions is quite essential. Applications of several methods and approaches have been done even though the results are not exhaustive.

2.2 Sawada-Kotera Equation

Hui *et al.* [16] developed nonlinear superposition formula used to construct Darboux and Backlund transformations for super symmetric Sawada-Kotera equation. Hui constructed periodic wave solutions using Hirota bilinear method basing on the Riemann theta function given by Fourier series representation of KdV-Sawada-Kotera equation.

Inc *et al.* [17] obtained many accurate and estimate solutions of fractional order Sawada-Kotera equation. They applied shifted modified Chebyshev wavelet technique and expansion method whereby such solutions were found through exponential, rational, trigonometric and hyperbolic functions. The procedure was reinforced by numerical data and graphical representation.

Inc *et al.* [17] used Hirota bilinear method and the ansatz approach to construct soliton solutions for Sawada-Kotera equation to obtain topological and non-topological and multi-soliton solutions. The obtained solutions enabled them to plot 2-D and 3-D figures using Mathematica 9.

Xiazhi *et al.* [38] derived nonlocal symmetries of the residual symmetry and spectral function symmetry of Sawada-Kotera equation from the truncated Painleve expansion method and Lax pair method. By localizing the nonlocal symmetries of the original system to the prolonged systems of local ones, transformations of Darboux and Backlund are obtained.

Olaniyi [26] considered time-fractional forms of the Sawada-Kotera equation and the Ito equation by calculating the approximate solutions in the form of series obtained by means q-Homotopy Analysis Method (q-HAM). Analytical methods were compared with with Modified Adomian decomposition method (MADM), homotopy perturbation method (HPM) due to the presence of fraction-factor. The estimated results were compared with the precise results. Numerical solutions were then obtained using Mathematica 8.

He and Geng [15] derived a sequential order of the new nonlinear evolution equations of the Sawada-Kotera equation where they introduced a 3×3 matrix spectral problem having two potentials. This was done with the help of lax pairs. They were also able to construct endless sequences of the conserved quantities of evolution equations.

Sh. Sadigh [35] solved Sawada-Kotera equation by means of the Adomian's decomposition technique. The other methods used were variational iteration and homotopy perturbation. He also applied the modified processes of the techniques used. The estimate results of the Sawada-Kotera equation was solved in the form of sequence whereby its components were calculated using recursive relation. The convergence of the proposed methods and the presence and distinctiveness of the results were verified. He studied a numerical model to determine the exactness of the used procedures.

Abdul-Majid [1] obtained multiple singular solutions for the Sawada-Kotera equation using the simple form of Hirota's bilinear method.

Dai and Liu [8] used Hirota bilinear method to solve the fifth order Sawada-Kotera equation in which the results showed the existence of various solutions of the equation which could be classified as one-soliton, periodic two-soliton and also singular periodic soliton solutions. These solutions gave the exact soliton solutions of the equation.

W.L and Yu-Kun [37] solved the Caudrey-Dodd-Gibbon-Sawada-Kotera equation by introducing a λ - modified equation. They used invariance property for the equation under crum transformation to derive a new Backlund transformation.

2.3 Time Fractional Partial Differential Equation

Khongorzul *et al.* [21] used Lie symmetry analysis in the study of time fractional partial differential equations which are termed to be nonlinear evolution systems where a classification of group invariant, infinitesimal symmetries, a complete group classification and the solutions were obtained. This was done by dividing it into two cases based on the function contained. Infinitesimal symmetries generated the dimension whereby in each case was greater than two hence presentation of the arrangements and one-dimensional optimal systems of the Lie algebras. They further obtained the reduced schemes equivalent to optimal systems and explicit set invariant results for each case.

Manoj and Karanjeet [23] presented Lie point symmetries to solve time-fractional Burgers' equation. The symmetries were used to transform the equation into an ordinary differential equation of fractional order which corresponded to the Erdelyi-Kober fractional derivative. An invariant subspace method was then used to provide an analytic solution.

Youwei [40] considered two classes of the general time-fractional Korteweg-de Vries equations (KdVs) where an orderly analysis to develop Lie point symmetries of the models were obtainable and comparison was done. This was done by transforming both equations to form a nonlinear ordinary differential equation consisting of different independent variable. A derivative that is equivalent to time-fractional in the condensed technique was called the Erdelyi-Kober fractional derivative. Gang *et al.* [12] used the Lie group analysis technique to find the invariance properties of the time fractional fifth-order KdV equation. They performed a procedural study for deriving Lie point symmetries of time fractional fifth-order KdV equation. They obtained reductions in symmetry and the vector fields of the fractional fifth-order KdV equation by means of point symmetry. They therefore provided some exact solutions using sub-equation method.

2.4 Linearized Differential Equation

Zablon and Sogomo [41] studied how to solve differential equations by symmetry groups for first order ODEs by exploring the possibility of averting the assumptions in applications of Lie groups to differential equation. They found out that solving the original ODE was much easier than getting the solutions of the linearized symmetry condition (the symbols ξ and η). By inspired presumption, or geometric perception, it was possible to determine a particular solution of the linearized symmetry condition which permitted the integration of the original equation. Solving differential equations involved some guesses and assumptions of the form of symmetry for a given differential equation using Lie group symmetry.

2.5 Nonlinear Helmholtz Equation

Sakkaravathi *et al.* [34] considered the nonlinear Helmholtz (NLH) equation where they described the beam transmission in a planar waveguide with Kerr-like nonlinearity considering the non-paraxial estimation. Using the optimal systems of one-dimensional sub algebras, they determined the Lie point symmetries of ordinary differential equations (ODEs) and their equivalent symmetry reductions by applying the Lie symmetry analysis. Their analysis revealed significant information that even if the original equation was non-integrable, its symmetry reductions were Painleve integrable. They analytically studied the solution sets of nonlinear ODEs by getting explicit travelling wave solutions that included single and symbiotic single wave solutions by constructing the integrals of motion by means of the adapted Prelle-Singer technique and also by carrying out an exhaustive numerical study of the reduced equations with the aim of obtaining multi-peak nonlinear wave trains. They did compare the symmetries in the standard nonlinear Schrodinger equation and that of the equation being studied whose symmetries were since existing in the literature.

2.6 Nonlinear Wave Equation

Islam *et al.* [18] implemented the $\exp(-\emptyset(\xi))$-expansion technique in constructing the precise traveling wave results for nonlinear evolution equations (NLEEs). They considered two model equations which played important function in nonlinear sciences, which are known to be the time regularized long wave (TRLW) equation and the Korteweg-de Vries (KdV) equation. They found trigonometric, rational, hyperbolic and also exponential explicit function solutions of the variables in the chosen equations. They realized that the used technique was quite effective and was virtually suitable for the aforesaid problems and subsequently for the other NLEEs which arises in engineering fields and those arising in mathematical physics.

Aminer [2] used Lie symmetry analysis technique to obtain the exact results of the fourth order nonlinear ordinary differential equation which was a one-dimensional wave equation. The approach was efficient because the results did not rely on both boundary and initial conditions and was not an approximation to the exact solution. Therefore, the study employed a systematic process of developing variational symmetries, infinitesimal transformations, generators, integrating factors, prolongations (extended transformations), adjoint-symmetries and the invariant transformations of the model in question. The method was meant to lower the order of the model from fourth to first order, and was then calculated to obtain the Lie symmetry result.

Bluman and Anco [5] obtained adjoint symmetries for the wave equation but they did not attain much since the variational symmetries and also true symmetries were not identified. They found all the integrating factors and their equivalent first integrals for any given scheme of ODEs. The obtained integrating factors were revealed to be all the results of both the adjoint scheme of the linearized scheme of ODEs and also a scheme that represented additional adjoint invariance conditions.

Fritz [11] dealt with Lie's theory for solving second-order quasilinear differential equations based on its symmetries applied for designing solution algorithms. He supplemented the Lie's original theory by different results that had been obtained after his death one hundred years ago. This was right above all of Janet's theory [19] for schemes of linear partial differential equations and of Loewy's theory [22] for decomposing linear differential equations into components of lowest order. The outcome allowed the formulation of the similarity problems that the were associated with Lie symmetries and mainly, determination of the function field in which the transformation functions act was considered as part of the problem.The equation that initially had to be solved determined the base field, i.e. the smallest field containing its coefficients. The fields that occurred later on in the solution process were extensions of the base field and were determined clearly. The study showed that a symmetric equation could be solved in closed form algorithmically by transforming into a canonical form equivalent to its symmetry type by Liouvillian transformation basing on the base field thus describing a solution algorithm. Computer algebra software on top of the type system ALL TYPES availed so as to make it easier to apply these algorithms to existing problems.

2.7 Nonlinear Beam Equation

Dingjiang *et al.* [9] studied a generalized nonlinear beam equation which were of second-order wave terms and fourth-order wave terms, that was prolonged from the classical beam equation occurring in the past procedures of travelling wave manner in the Golden Gate Bridge in San Francisco using Lie symmetry analysis. They used the equivalence transformation group theory to carry out a total Lie symmetry group categorization. They separated out from the classification solutions the investigated Lie symmetry reduction of a nonlinear beam-like model thus by ways of the reductions and representative calculation, some classes of precise solutions, as well as single wave results, triangular sporadic wave results and normal results of the nonlinear beam-like equations were composed.

2.8 Nonlinear ODEs

Oliveri [27] presented Lie symmetry analysis of differential equations which provided a strong and essential outline to the utilization of orderly methods that leads to integrating by quadrature (or at least to lowering the order) of ordinary differential equations, to the obtaining of constant results of problems containing initial and boundary values, to the deduction of conservation laws, to the creation of associations among diverse differential equations that could be equal. Review of some familiar solutions of Lie group analysis, including the current contributions concerned with the conversion of differential equations to corresponding systems that are important to study related problems was done.

George and Gregory [13] presented a theory for determining new symmetries for ODEs which lead to an orderly reduction of the order of a differential equation. They used a Backlund transformation to work out the Lie symmetries of a differential equation thus inducing different symmetries of the known equation which were neither of contact, Lie nor of Lie- Backlund form. They obtained new symmetries and the equivalent new critical outcomes for a set of ordinary differential equations occurring from nonlinear diffusion.

2.9 Nonlinear PDEs

Roman *et al.* [33] reviewed on finding a precise results of a set of reaction-diffusion-convection equations consisting of exponential nonlinearities and through these technique they the were able to look for Lie and Q-conditional also called non-classical symmetries. They used two different algorithms to derive a total Lie symmetry arrangement of the class so as to illustrate that the solution depended basically on the type of correspondence transformations that are useful for the arrangement. They also presented a total explanation of Q-conditional symmetries for PDEs. It was revealed that all the renowned solutions for the equations with exponential nonlinearities followed as exact cases from the solutions resulting for the class of similar equations. They constructed accurate results of the related equations by obtaining the symmetries that were then compared with those that were established by means of different techniques and eventually presented the use of the exact results for finding the solutions of boundary-value problems obtained in population dynamics.

Roman and Maksym [32] studied a simplified Keller–Segel model by applying Lie symmetry technique. They illustrated that a (1 + 2)-dimensional Keller–Segel form scheme, jointly including the rightly-specified boundary and initial values was invariant with regards to infinite-dimensional Lie algebras. They presented Lie symmetry arrangement of the Cauchy scheme which depended on the initial and boundary condition which were then used in reducing the order of the problem to obtain a (1 + 1)-dimensional scheme. They further, verified that the Cauchy method for the (1 + 1)-dimensional simplified scheme could be linearized then answered in an explicit form by constructing accurate results of various (1 + 1)-dimensional problems. They also established, motivated restrictions and derived

Lie symmetry categorization of the (1 + 2)-dimensional Neumann scheme for the simplified Keller–Segel system so as to get a single solution. Since Lie symmetry of boundary-value model depend basically on geometry of the area, which the problem was formulate for, they examined the realistic domains. Reduction of the Neumann scheme on a band was determined by means of the symmetries obtained so as to find an accurate result of a nonlinear two-dimensional Neumann system on a set period.

Andronikos *et al.* [4] analyzed two sets of (1+2)(1+2) evolution equations that were of particular concern in Financial Mathematics, such as the model for the Two-factor Commodities schemes and the Two-dimensional Black-Scholes model using Lie Symmetry Analysis. They studied problems for the case that were independent and those whose parameters of the equation were indefinite function of time. Thus in the independent Black-Scholes Equation, they established that their symmetry was maximal hence the equation could be reduced to the (1+2)(1+2)Classical Heat Equation. It was different in the example for the dependent equation whereby the amount of symmetries was submaximal. Considering the two-factor equation, it was found out that the quantity of symmetries was submaximal in independent and also in dependent situations. When the resulting symmetries were applied to reduce the order of each of the schemes to obtain a (1+1)(1+1) equation, the resultant scheme was of greatest symmetry and hence equal to the (1+2)(1+2) Classical Heat Equation.

Roman and John [31] proposed an innovative description of restricted invariance for boundary value schemes which involved an extensive series of boundary conditions. It was revealed that further descriptions were workable in finding Lie symmetries of boundary value schemes through normal boundary conditions which followed a specific examples from definitions. They established that the study was applicable to the nonlinear problems since they were able to solve simple examples that were arising in systems. They realized a thriving use of the description for the Lie and restricted symmetry arrangement of a set of nonlinear boundary value schemes that were of (1 + 2)-dimension which were administered by the nonlinear diffusion model in a semi-infinite field. When the scheme in question with non-diasppearing change on the boundary admitted extra Lie symmetry mechanisms linked to when $k \neq -2$, it was established that there was a unique model, $k \neq -2$, for the power diffusivity u^k thus were useful in reducing the nonlinear systems with power diffusivity u^k and an unvarying non-zero change on the boundary which was ordinary in uses and described an extensive series of phenomenon to (1 + 1)-dimensional systems so as to reveal the applicability of the resulting symmetries. After analyzing structures and properties of the problems obtained, they presented a number of solutions representing how Lie invariance of the boundary value scheme in the study depended on the geometry of the field.

Aminus [3] considered Laplace equation on surfaces of revolution and discussed the symmetry algebra based on classical Lie symmetry theory. Symmetry reductions were applied in order to obtain new harmonic functions on surfaces of revolution using the Lie point symmetries.

Juan *et al.* [20] found explicit results of nonlinear Schrodinger equations that have spatially inhomogeneous nonlinearities by means of Lie group theory and also canonical transformation. They presented the general theory, thus used it to solve diverse models and used the qualitative theory of vibrant schemes to find various properties of those results.

Popovych [30] discussed the reduction operators of the linear parabolic partial differential equations and provided theoretical results on some transformation and reductions for determining equations. His main result was a series of 'no-go' theorems concerning symmetries that did not lead to new reductions.

Oduor [25] solved Burgers equation,$u_t - uu_x = \lambda u_x$ which is a non-linear PDE arising from model study of turbulence and shock wave theory. He determined all the Lie groups admitted by Burgers equation and used symmetry transformations to establish all the global solutions corresponding to each Lie group admitted by the equation.

Nucci *et al.* [24] used role of symmetries to solve differential equations, hence showing the solutions on the use of classical lie point symmetries in solving equations involving epidemiology of nutrition and meteorology. The iteration of the scheme yielded new non-linear equations that inherited the lie symmetry algebra of the specified system. The invariant results of the non-linear equations formed gave new results of the initial equation.

Omolo [29] used lie symmetry analysis of differential equations in solving nonlinear differential equations. He gave a stability approach to exact solutions of non-linear PDEs provided by the symmetry groups.

Despite the fact that so many scholars did much work in Sawada-Kotera equation, the solutions obtained were approximate to the exact solutions. To fill the gap, Lie Symmetry analysis provided exact symmetry solutions of Sawada-Kotera equation.

CHAPTER THREE

METHODOLOGY

3.1 Introduction

This chapter precisely illustrate the techniques and procedures that were applied in solving equation (1.1).

3.2 Groups of Transformation and Infinitesimal Transformations

The study generated infinitesimal generators, infinitesimal transformations and the groups in which the equation admitted.

The groups of transformations required were of the form;

$$x^* = X(x,t,u;\varepsilon) \tag{3.1a}$$

$$t^* = T(x,t,u;\varepsilon) \tag{3.1b}$$

$$u^* = U(x,t,u;\varepsilon) \tag{3.1c}$$

and their corresponding infinitesimal transformations α, β, λ in which

$$\alpha(x,t,u) = \frac{\partial X(x,t,u;\varepsilon)}{\partial x}|_{\varepsilon=0} \tag{3.2a}$$

$$\beta(x,t,u) = \frac{\partial T(x,t,u;\varepsilon)}{\partial t}|_{\varepsilon=0} \tag{3.2b}$$

$$\lambda(x,t,u) = \frac{\partial U(x,t,u;\varepsilon)}{\partial u}|_{\varepsilon=0} \tag{3.2c}$$

3.3 Using Lie's Integrating Factor

The study uses the solutions of adjoint symmetries of the linearized PDE which act as integrating factor by applying theorems that show the link between infinitesimal symmetries and integrating factors. Such theorems are stated as follows:

Theorem 3.1

Consider a first order differential equation represented in the form of symmetry as

$$Q(x,y)dx+R(x,y)dy = 0 \tag{3.3}$$

Lie showed that (3.3) admits a one- parameter group P with the infinitesimal generator expressed as

$$G = \beta(x,y)\frac{\partial}{\partial x}+\lambda(x,y)\frac{\partial}{\partial y} \tag{3.4}$$

where β and λ are functions of x and y only.

Thus equation (3.4) is a symmetry for equation (3.3) and

$$\beta = (\beta Q+\lambda R)^{-1} \tag{3.5}$$

is called Lie's integrating factor for equation (3.3) provided that

$\beta Q + \lambda R \neq 0$

Example 3.1

Considering the Riccati equation of the form

$y' + y^2 - \frac{2}{x^2} = 0$

We re-write it in the form of equation (3.1) to obtain

$dy + (y^2 - \frac{2}{x^2})dx$

Substituting $\beta = x, y = -y, Q = y^2 - \frac{2}{x^2}$ and $R = 1$ we obtain the integrating factor

$\phi = \frac{x}{x^2y^2-xy-2}$

On multiplying the Riccati equation by this integrating factor we obtain

$\frac{xdy+(xy^2-\frac{2}{x})dx}{x^2y^2-xy-2} = 0$

We then re-write in the following form for integration to be done

$$= \frac{xdy+ydx}{x^2y^2-xy-2} + \frac{dx}{x}$$

$$= d(lnx + \frac{1}{3}ln\frac{xy-2}{xy+1}) = 0$$

and finally we integrate to obtain

$\frac{xy-2}{xy+1} = \frac{k}{x^3}$

$\Rightarrow x^3 \frac{xy-2}{xy+1} = k$

thus to solve for y, we find the solution of the Ricatti equation as

$y = \frac{2x^3+k}{x(x^3-k)}$

where k- constant.

3.4 Invariant Transformation of Differential Equations

The study applied infinitesimal transformations in constructing the solutions of differential equations.

This was actualized by considering systems of differential equations and showing the infinitesimal criterion of their invariance.

The results of the algorithm were used to find out the infinitesimal generators represented by the equation.

According to Olver [28], invariant surfaces of the corresponding Lie group of point transformations lead to invariant solutions (similarity solutions).The solutions were obtained by solving partial differential equations with fewer independent variables than the given PDE.

First, we consider a k^{th} order differential equation of the form

$$F(x, u, u_1, u_2, u_3, ..., u_k) = 0 \tag{3.6}$$

where $x = (x_1, x_2, x_3, ..., x_n)$ denotes n independent variables, u_j denotes the set of coordinates corresponding to all the j^{th} order partial derivatives with respect to x . We assume that the Partial Differential Equation (3.6) can be written in solvable form in terms of some k^{th} order partial derivative of u .

$$F(x, u, u_1, u_2, u_3, ..., u_k) = u_{i_1 i_2 i_3 i_4 i_5 ....i_j} - f(x, u, u_1, u_2, u_3, ..., u_k) = 0 \tag{3.7}$$

where $f(x, u, u_1, u_2, u_3, ..., u_k)$ does not depend on $u_{i_1 i_2 i_3 i_4 i_5 ....i_j}$.

We now give a criterion for the invariance of a partial differential equation by stating the theorem below [6].

Theorem 3.2

Let $F_a(x, u^{(k)}) = 0$ be a non-degenerate system of differential equations.

Let $V = \alpha_i(x,u)\frac{\partial}{\partial x_i} + \lambda(x,u)\frac{\partial}{\partial u}$ be the infinitesimal generator of the one-parameter Lie group of transformations given as

$x^* = X(x, u; \epsilon)$ (3.8)

$u^* = U(x, u; \epsilon)$ (3.9)

and let

$V^{(k)} = \alpha_i(x,u)\frac{\partial}{\partial x_i} + \lambda(x,u)\frac{\partial}{\partial u} + \lambda_i^{(1)}(x,u,u_1)\frac{\partial}{\partial u_1} + ... + \lambda^{(k)}_{i_1 i_2 i_3 ... i_k}(x, u, u_1, u_2, u_3, ...u_k)$
$\frac{\partial}{\partial u_{i_1 i_2 i_3 i_4 i_5 .... i_k}}$ (3.10)

be the corresponding k^{th} extended infinitesimal generator where

$\lambda_i^{(l)} = \lambda(x, y)$

is given by

$\lambda_i^{(l)} = D_i\lambda - (D_i\alpha_j)u_j, i = 1, 2, 3, ..., n;$ (3.11)

and $\lambda^{(j)}_{i_1 i_2 i_3 i_4 i_5 ... i_j}$ is given by

$\lambda^{(k)}_{i_1 i_2 i_3 i_4 i_5 ... i_k} = D_{i_k}\lambda^{(k-1)}_{i_1 i_2 i_3 i_4 i_5 ... i_k - 1} - (D_{i_k}\lambda_j)u_{i_1 i_2 i_3 i_4 ... i_{k-1}}$ (3.12)

$i_j = 1, 2, 3, ..., n$ for, $j = 1, 2, 3, ..., k$ with $k = 1, 2, 3, ...$ in terms of $(\alpha(x,u), \lambda(x,u))$.

Then one-parameter Lie group of transformations (3.8) and (3.9) is admitted by the partial differential equation (3.6) if and only if

$V^{(k)}[F_a(x, u, u_1, u_2, u_3, ..., u_k)] = 0, a = 1, 2, 3, ..., l$ (3.13)

Whenever

$F(x, u^{(k)}) = 0$

3.5 Lie Point Symmetries

The study describe Lie point symmetry as a point that depends continuously on at least one parameter since the parameters can vary over a set of nonzero measure. The Lie point symmetries of PDEs are represented in the form

$G = \alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial t} + \lambda\frac{\partial}{\partial u}$ (3.14) where α, β and λ are functions of x, t and u only.

For us to be able to apply a point transformation on our equation there is need to know how the derivatives transform in the infinitesimal transformation

$\overline{x} = x + \varepsilon\alpha(x,t,u) + 0(\varepsilon^2)$

$\overline{t} = t + \varepsilon\beta(x,t,u) + 0(\varepsilon^2)$

$$\overline{u} = u + \varepsilon\lambda(x,t,u) + 0(\varepsilon^2) \tag{3.15}$$

whose generator is known to be

$$G^{[0]} = \alpha(x,t,u)\frac{\partial}{\partial x} + \beta(x,t,u)\frac{\partial}{\partial t} + \lambda(x,t,u)\frac{\partial}{\partial u} \tag{3.16}$$

We also find the first, second, third, fourth and fifth derivatives of $\overline{x}$ and $\overline{t}$ then find the extensions/prolongations of the generator G.

The prolongations of the generator from the first to the fifth are: [28]

$$G^{[1]} = G^{[0]} + \lambda^t\frac{\partial}{\partial u_t} + \lambda^x\frac{\partial}{\partial u_x} \tag{3.17}$$

$$G^{[2]} = G^{[1]} + \lambda^{tt}\frac{\partial}{\partial u_{tt}} + \lambda^{tx}\frac{\partial}{\partial u_{tx}} + \lambda^{xx}\frac{\partial}{\partial u_{xx}} \tag{3.18}$$

$$G^{[3]} = G^{[2]} + \lambda^{ttt}\frac{\partial}{\partial u_{ttt}} + \lambda^{ttx}\frac{\partial}{\partial u_{ttx}} + \lambda^{txx}\frac{\partial}{\partial u_{txx}} + \lambda^{xxx}\frac{\partial}{\partial u_{xxx}} \tag{3.19}$$

$$G^{[4]} = G^{[3]} + \lambda^{tttt}\frac{\partial}{\partial u_{tttt}} + \lambda^{tttx}\frac{\partial}{\partial u_{tttx}} + \lambda^{ttxx}\frac{\partial}{\partial u_{ttxx}} + \lambda^{txxx}\frac{\partial}{\partial u_{txxx}} + \lambda^{xxxx}\frac{\partial}{\partial u_{xxxx}} \tag{3.20}$$

$$G^{[5]} = G^{[4]} + \lambda^{ttttt}\frac{\partial}{\partial u_{ttttt}} + \lambda^{ttttx}\frac{\partial}{\partial u_{ttttx}} + \lambda^{tttxx}\frac{\partial}{\partial u_{tttxx}} + \lambda^{ttxxx}\frac{\partial}{\partial u_{ttxxx}} + \lambda^{txxxx}\frac{\partial}{\partial u_{txxxx}} + \lambda^{xxxxx}\frac{\partial}{\partial u_{xxxxx}} \tag{3.21}$$

Where the terms$\lambda^t, \lambda^x, \lambda^{xx}, \lambda^{xxx}$... are coefficients and are generated and expressed in terms of partial derivatives as shown below

$\lambda^x = \frac{\partial\lambda}{\partial x} + u'\frac{\partial\lambda}{\partial u}$ { from $d(\lambda = \frac{\partial\lambda}{\partial x}dx + (\frac{\partial\lambda}{\partial u})du\}$ hence

$$D_x(\lambda) = \lambda_x + u_x\lambda_u : \lambda(x,t,u) \tag{3.22}$$

$\alpha^x = \frac{\partial\alpha}{\partial x} + u'\frac{\partial\alpha}{\partial u}$ { from $d(\alpha = \frac{\partial\alpha}{\partial x}dx + (\frac{\partial\alpha}{\partial u})du\}$ hence

$$D_x(\alpha) = \alpha_x + u_x\alpha_u : \alpha(x,t,u) \tag{3.23}$$

$\beta^x = \frac{\partial\beta}{\partial x} + u'\frac{\partial\beta}{\partial u}$ { from $d(\beta = \frac{\partial\beta}{\partial x}dx + (\frac{\partial\beta}{\partial u})du\}$ hence

$$D_x(\beta) = \beta_x + u_x\beta_u : \beta(x,t,u) \tag{3.24}$$

$\lambda^t = \frac{\partial\lambda}{\partial t} + u'\frac{\partial\lambda}{\partial u}$ { from $d(\lambda = \frac{\partial\lambda}{\partial t}dt + (\frac{\partial\lambda}{\partial u})du\}$ hence

$$D_t(\lambda) = \lambda_t + u_t\lambda_u) \tag{3.25}$$

$\alpha^t = \frac{\partial\alpha}{\partial t} + u'\frac{\partial\alpha}{\partial u}$ { from $d(\alpha = \frac{\partial\alpha}{\partial t}dt + (\frac{\partial\alpha}{\partial u})du\}$ hence

$$D_t(\alpha) = \alpha_t + u_t\alpha_u) \tag{3.26}$$

$\beta^t = \frac{\partial\beta}{\partial t} + u'\frac{\partial\beta}{\partial u}$ { from $d(\beta = \frac{\partial\beta}{\partial t}dt + (\frac{\partial\beta}{\partial u})du\}$ hence

$$D_x(\beta) = \beta_t + u_t\beta_u \tag{3.27}$$

$$\begin{aligned}
\lambda'' &= \tfrac{d}{dx}(\tfrac{\partial\lambda}{\partial x} + u'\tfrac{\partial\lambda}{\partial u}) + \tfrac{d}{dx}(\tfrac{\partial\lambda}{\partial x} + u'\tfrac{\partial\lambda}{\partial u})u' \\
&= \tfrac{\partial^2\lambda}{\partial x^2} + u'\tfrac{\partial^2\lambda}{\partial u\partial x} + u''\tfrac{\partial\lambda}{\partial u} + u'\tfrac{\partial^2\lambda}{\partial u\partial x} + u'^2\tfrac{\partial^2\lambda}{\partial u^2} + 0 \\
&= \tfrac{\partial^2\lambda}{\partial x^2} + 2u'\tfrac{\partial^2\lambda}{\partial u\partial x} + u'^2\tfrac{\partial^2\lambda}{\partial u^2} + u''\tfrac{\partial\lambda}{\partial u} \text{ hence}
\end{aligned}$$

$$D_x^2(\lambda) = \lambda_{xx} + 2u_x\lambda_{ux} + u_{xx}\lambda_u + u_x^2\lambda_{uu} \tag{3.28}$$

$$\begin{aligned}
\alpha'' &= \tfrac{d}{dx}(\tfrac{\partial\alpha}{\partial x} + u'\tfrac{\partial\alpha}{\partial u}) + \tfrac{d}{dx}(\tfrac{\partial\alpha}{\partial x} + u'\tfrac{\partial\alpha}{\partial u})u' \\
&= \tfrac{\partial^2\alpha}{\partial x^2} + u'\tfrac{\partial^2\alpha}{\partial u\partial x} + u''\tfrac{\partial\alpha}{\partial u} + u'\tfrac{\partial^2\alpha}{\partial u\partial x} + u'^2\tfrac{\partial^2\alpha}{\partial u^2} + 0 \\
&= \tfrac{\partial^2\alpha}{\partial x^2} + 2u'\tfrac{\partial^2\alpha}{\partial u\partial x} + u'^2\tfrac{\partial^2\alpha}{\partial u^2} + u''\tfrac{\partial\alpha}{\partial u} \text{ hence}
\end{aligned}$$

$$D_x^2(\alpha) = \alpha_{xx} + 2u_x\alpha_{ux} + u_{xx}\alpha_u + u_x^2\alpha_{uu} \tag{3.29}$$

$$\begin{aligned}
\beta'' &= \tfrac{d}{dx}(\tfrac{\partial\beta}{\partial x} + u'\tfrac{\partial\beta}{\partial u}) + \tfrac{d}{dx}(\tfrac{\partial\beta}{\partial x} + u'\tfrac{\partial\beta}{\partial u})u' \\
&= \tfrac{\partial^2\beta}{\partial x^2} + u'\tfrac{\partial^2\beta}{\partial u\partial x} + u''\tfrac{\partial\beta}{\partial u} + u'\tfrac{\partial^2\beta}{\partial u\partial x} + u'^2\tfrac{\partial^2\beta}{\partial u^2} + 0 \\
&= \tfrac{\partial^2\beta}{\partial x^2} + 2u'\tfrac{\partial^2\beta}{\partial u\partial x} + u'^2\tfrac{\partial^2\beta}{\partial u^2} + u''\tfrac{\partial\beta}{\partial u} \text{ hence}
\end{aligned}$$

$$D_x^2(\beta) = \beta_{xx} + 2u_x\beta_{ux} + u_{xx}\beta_u + u_x^2\beta_{uu} \tag{3.30}$$

$\lambda''' = \frac{d}{dx}(\frac{\partial^2\lambda}{\partial x^2} + 2u'\frac{\partial^2\lambda}{\partial u\partial x} + u'^2\frac{\partial^2\lambda}{\partial u^2} + u''\frac{\partial\lambda}{\partial u}) + u'\frac{d}{du}(\frac{\partial^2\lambda}{\partial x^2} + 2u'\frac{\partial^2\lambda}{\partial u\partial x} + u'^2\frac{\partial^2\lambda}{\partial u^2} + u''\frac{\partial\lambda}{\partial u})$

$= \frac{\partial^3\lambda}{\partial x^3} + 2u'\frac{\partial^3\lambda}{\partial u\partial x^2} + 2u''\frac{\partial^2\lambda}{\partial u\partial x} + u''\frac{\partial^2\lambda}{\partial u\partial x} + u''\frac{\partial^2\lambda}{\partial u\partial x} + u'''\frac{\partial\lambda}{\partial u} + u'^2\frac{\partial^3\lambda}{\partial u^2\partial x} + 2u'u''\frac{\partial^2\lambda}{\partial u^2} +$
$u'\frac{\partial^3\lambda}{\partial u\partial x^2} + 2u'^2\frac{\partial^3\lambda}{\partial u^2\partial x} + 0 + u'u''\frac{\partial^2\lambda}{\partial u^2} + 0 + u'^3\frac{\partial^3\lambda}{\partial u^3} + 0$

$= \frac{\partial^3\lambda}{\partial x^3} + 3u'\frac{\partial^3\lambda}{\partial u\partial x^2} + 3u''\frac{\partial^2\lambda}{\partial u\partial x} + u'''\frac{\partial\lambda}{\partial u} + 3u'^2\frac{\partial^3\lambda}{\partial u^2\partial x} + 3u'u''\frac{\partial^2\lambda}{\partial u^2} + u'^3\frac{\partial^3\lambda}{\partial u^3}$

Hence

$$D_x^3\lambda = \lambda_{xxx} + 3u_x\lambda_{uxx} + 3u_{xx}\lambda_{ux} + u_{xxx}\lambda_u + 3u_x^2\lambda_{uux} + 3u_xu_{xx}\lambda_{uu} + u_x^3\lambda_{uuu} \tag{3.31}$$

$\lambda^{(4)} = \frac{d}{dx}(\frac{\partial^3\lambda}{\partial x^3} + 3u'\frac{\partial^3\lambda}{\partial u\partial x^2} + 3u''\frac{\partial^2\lambda}{\partial u\partial x} + u'''\frac{\partial\lambda}{\partial u} + 3u'^2\frac{\partial^3\lambda}{\partial u^2\partial x} + 3u'u''\frac{\partial^2\lambda}{\partial u^2} + u'^3\frac{\partial^3\lambda}{\partial u^3}) +$
$u'\frac{d}{du}(\frac{\partial^3\lambda}{\partial x^3} + 3u'\frac{\partial^3\lambda}{\partial u\partial x^2} + 3u''\frac{\partial^2\lambda}{\partial u\partial x} + u'''\frac{\partial\lambda}{\partial u} + 3u'^2\frac{\partial^3\lambda}{\partial u^2\partial x} + 3u'u''\frac{\partial^2\lambda}{\partial u^2} + u'^3\frac{\partial^3\lambda}{\partial u^3})$

$= \frac{\partial^4\lambda}{\partial x^4} + 3u'\frac{\partial^4\lambda}{\partial u\partial x^3} + 3u''\frac{\partial^3\lambda}{\partial u\partial x^2} + 3u''\frac{\partial^3\lambda}{\partial u\partial x^2} + 3u'''\frac{\partial^2\lambda}{\partial u\partial x} + u'''\frac{\partial^2\lambda}{\partial u\partial x} + u^{(4)}\frac{\partial\lambda}{\partial u} + 3u'^2\frac{\partial^4\lambda}{\partial u^2\partial x^2} +$
$6u'u''\frac{\partial^3\lambda}{\partial u^2\partial x} + 3u'u''\frac{\partial^3\lambda}{\partial u^2\partial x} + 3u'u'''\frac{\partial^2\lambda}{\partial u^2} + 3u''^2\frac{\partial^2\lambda}{\partial u^2} + u'^3\frac{\partial^4\lambda}{\partial u^3\partial x} + 3u'^2u''\frac{\partial^3\lambda}{\partial u^3} + u'\frac{\partial^4\lambda}{\partial u\partial x^3} +$
$3u'^2\frac{\partial^4\lambda}{\partial u^2\partial x^2} + 3u'u''\frac{\partial^3\lambda}{\partial u^2\partial x} + u'u'''\frac{\partial^2\lambda}{\partial u^2} + 3u'^3\frac{\partial^4\lambda}{\partial u^3\partial x} + 3u'^{2u''\frac{\partial^3\lambda}{\partial u^3}+u'^4\frac{\partial^4\lambda}{\partial u^4}}$

$= \frac{\partial^4\lambda}{\partial x^4} + 4u'\frac{\partial^4\lambda}{\partial u\partial x^3} + 6u''\frac{\partial^3\lambda}{\partial u\partial x^2} + 4u'''\frac{\partial^2\lambda}{\partial u\partial x} + u^{(4)}\frac{\partial\lambda}{\partial u} + 3u'^2\frac{\partial^4\lambda}{\partial u^2\partial x^2} + 9u'u''\frac{\partial^3\lambda}{\partial u^2\partial x} +$
$4u'u'''\frac{\partial^2\lambda}{\partial u^2} + 3u''^2\frac{\partial^2\lambda}{\partial u^2} + 4u'^3\frac{\partial^4\lambda}{\partial u^3\partial x} + 6u'^2u''\frac{\partial^3\lambda}{\partial u^3} + 3u'^2\frac{\partial^4\lambda}{\partial u^2\partial x^2} + 3u'u''\frac{\partial^3\lambda}{\partial u^2\partial x} + u'^4\frac{\partial^4\lambda}{\partial u^4}$

Hence we have

$D_x^4\lambda = \lambda_{xxxx} + 4u_x\lambda_{uxxx} + 6u_{xx}\lambda_{uxx} + 4u_{xxx}\lambda_{ux} + u_{xxxx}\lambda_u + 6u_x^2\lambda_{uuxx} + 12u_xu_{xx}\lambda_{uuxx} +$

$3u_{xx}^2\lambda_{uu} + 4u_xu_{xxx}\lambda_{uu} + 4u_x^3\lambda_{uuux} + 6u_x^2u_{xx}\lambda_{uuu} + u_x^4\lambda_{uuuu}$ (3.32)

$\lambda^{(5)} = \frac{d}{dx}[\frac{\partial^4\lambda}{\partial x^4} + 4u'\frac{\partial^4\lambda}{\partial u\partial x^3} + 6u''\frac{\partial^3\lambda}{\partial u\partial x^2} + 4u'''\frac{\partial^2\lambda}{\partial u\partial x} + u^{(4)}\frac{\partial\lambda}{\partial} + 6u'^2\frac{\partial^4\lambda}{\partial u^2\partial x^2} + 12u'u''\frac{\partial^3\lambda}{\partial u^2\partial x}) +$
$3u''^2\frac{\partial^2\lambda}{\partial u^2} + 4u'u'''\frac{\partial^2\lambda}{\partial u^2} + 4u'^3\frac{\partial^4\lambda}{\partial u^3\partial x} + 6u'^2u''\frac{\partial^3\lambda}{\partial u^3} + u'^4\frac{\partial^4\lambda}{\partial u^4}] + u'\frac{d}{du}[\frac{\partial^4\lambda}{\partial x^4} + 4u'\frac{\partial^4\lambda}{\partial u\partial x^3} + 6u''\frac{\partial^3\lambda}{\partial u\partial x^2} +$
$4u'''\frac{\partial^2\lambda}{\partial u\partial x} + u^{(4)}\frac{\partial\lambda}{\partial} + u^{(4)}\frac{\partial\lambda}{\partial u} + 6u'^2\frac{\partial^4\lambda}{\partial u^2\partial x^2} + 12u'u''\frac{\partial^3\lambda}{\partial u^2\partial x} + 3u''^2\frac{\partial^2\lambda}{\partial u^2} + 4u'u'''\frac{\partial^2\lambda}{\partial u^2} + 4u'^3\frac{\partial^4\lambda}{\partial u^3\partial x} +$
$6u'^2u''\frac{\partial^3\lambda}{\partial u^3}] + u'^4\frac{\partial^4\lambda}{\partial u^4}]$

$= \frac{\partial^5\lambda}{\partial x^5} + 4u'\frac{\partial^5\lambda}{\partial u\partial x^4} + 4u''\frac{\partial^4\lambda}{\partial u\partial x^3} + 6u''\frac{\partial^4\lambda}{\partial u\partial x^3} + 6u'''\frac{\partial^3\lambda}{\partial u\partial x^2} + 4u'''\frac{\partial^3\lambda}{\partial u\partial x^2} + 4u^{(4)}\frac{\partial^2\lambda}{\partial u\partial x} +$
$u^{(4)}\frac{\partial^2\lambda}{\partial u\partial x} + u^{(5)}\frac{\partial\lambda}{\partial u} + 6u'^2\frac{\partial^5\lambda}{\partial u^2\partial x^3} + 12u'u''\frac{\partial^4\lambda}{\partial u^2\partial x^2} + 12u\prime u''\frac{\partial^4\lambda}{\partial u^2\partial x^2} + 12u''^2\frac{\partial^3\lambda}{\partial u^2\partial x} + 12u\prime u'''\frac{\partial^3\lambda}{\partial u^2\partial x} +$
$4u'u'''\frac{\partial^3\lambda}{\partial u^2\partial x} + 4u''u'''\frac{\partial^2\lambda}{\partial u^2} + 4u'u^{(4)}\frac{\partial^2\lambda}{\partial u^2} + 3u''^2\frac{\partial^3\lambda}{\partial u^2\partial x} + 6u''u'''\frac{\partial^2\lambda}{\partial u^2} + 6u'^2u''\frac{\partial^4\lambda}{\partial u^3\partial x} + 12u'u''^2\frac{\partial^3\lambda}{\partial u^3} +$
$6u'^2u'''\frac{\partial^3\lambda}{\partial u^3} + 4u'^3\frac{\partial^5\lambda}{\partial u^3\partial x^2} + 12u\prime 2u''\frac{\partial^4\lambda}{\partial u^3\partial x} + u'^4\frac{\partial^5\lambda}{\partial u^4\partial x} + 4u'^3u''\frac{\partial^4\lambda}{\partial u^4} + u'\frac{\partial^5\lambda}{\partial u^2\partial x^4} + 4u'^2\frac{\partial^5\lambda}{\partial u^2\partial x^3} +$
$6u'u''\frac{\partial^4\lambda}{\partial u^2\partial x^2} + 4u'u'''\frac{\partial^3\lambda}{\partial u^2\partial x} + u'u^{(4)}\frac{\partial^2\lambda}{\partial u^2} + 6u'^3\frac{\partial^5\lambda}{\partial u^3\partial x^2} + 12u'^2u''\frac{\partial^4\lambda}{\partial u^3\partial x} + 4u'^2u'''\frac{\partial^3\lambda}{\partial u^3} +$
$3u'u''^2\frac{\partial^3\lambda}{\partial u^3} + 6u'^3u''\frac{\partial^4\lambda}{\partial u^4} + 4u'^4\frac{\partial^5\lambda}{\partial u^4\partial x} + u'^5\frac{\partial^5\lambda}{\partial u^5}$

$= \frac{\partial^5\lambda}{\partial x^5} + 5u'\frac{\partial^5\lambda}{\partial u\partial x^4} + 10u''\frac{\partial^4\lambda}{\partial u\partial x^3} + 10u'''\frac{\partial^3\lambda}{\partial u\partial x^4} + 5u^{(4)}\frac{\partial^2\lambda}{\partial u\partial x} + u^{(5)}\frac{\partial\lambda}{\partial u} + 10u'^2\frac{\partial^5\lambda}{\partial u^2\partial x^3} +$
$30u'u'''\frac{\partial^4\lambda}{\partial u^2\partial x^2} + 15u''^2\frac{\partial^3\lambda}{\partial u^2\partial x} + 20u'u'''\frac{\partial^3\lambda}{\partial u^2\partial x} + 10u''u'''\frac{\partial^2\lambda}{\partial u^2} + 5u'u^{(4)}\frac{\partial^4\lambda}{\partial u^2} + 30u'^2u''\frac{\partial^4\lambda}{\partial u^3\partial x} +$
$15u'u''2\frac{\partial^5\lambda}{\partial u^4} + 10u'^2u'''\frac{\partial^3\lambda}{\partial u^3} + 10u'^3\frac{\partial^5\lambda}{\partial u^3\partial x^2} + 5u^4\frac{\partial^5\lambda}{\partial u^4\partial x} + 10u'^3u''\frac{\partial^4\lambda}{\partial u^4} + u'^5\frac{\partial^5\lambda}{\partial u^5}$

Hence

$D_x^5(\lambda) = \lambda_{xxxxx} + 5u_x\lambda_{uxxxx} + 10u_{xx}\lambda_{uxxx} + 10u_{xxx}\lambda_{uxx} + 5u_{xxxx}\lambda_{ux} + u_{xxxxx}\lambda_u +$
$10u_{x^2}\lambda_{uuxxx} + 30u_xu_{xx}\lambda_{uuxx} + 15u_{xx^2}\lambda_{uux} + 20u_xu_{xxx}\lambda_{uux} + 10u_{xx}u_{xxx}\lambda_{uu} + 5u_xu_{xxxx}\lambda_{uu} +$
$10u_{x^3}\lambda_{uuuxx} + 30u_{x^2}u_{xx}\lambda_{uuux} + 15u_xu_{xx^2}\lambda_{uuu} + 10u_{x^2}u_{xxx}\lambda_{uuu} + 5u_{x^4}\lambda_{uuuux} + 10u_{x^3}u_{xx}\lambda_{uuuu} +$
$u_{x^5}\lambda_{uuuuu}$ (3.33)

The terms $\lambda^t, \lambda^x, \lambda^{xx}, \lambda^{xxx}$ and λ^{xxxxx} are the coefficients and are generated and expressed in terms of partial derivatives as shown below

$\lambda^t = D_t\lambda - u_tD_t\beta - u_xD_t\alpha$
$= \lambda_t + u_t\lambda_u - u_t\beta_t - u_x\alpha_t - u_xu_t\alpha_u - u_t^2\beta_u$
$= \lambda_t + u_t(\lambda_u - \beta_t) - u_x\alpha_t - u_xu_t\alpha_u - u_t^2\beta_u$ (3.34)

$\lambda^x = D_x\lambda - u_xD_x\alpha - u_tD_x\beta$
$= \lambda_x + u_x\lambda_u - u_x\alpha_x - u_x^2\alpha_u - u_t\beta_x - u_xu_t\beta_u$
$= \lambda_x + u_x(\lambda_u - \alpha_x) - u_x^2\alpha_u - u_t\beta_x - u_xu_t\beta_u$ (3.35)

$\lambda^{tt} = D_t^2(\lambda -_x -_t) + \alpha u_{xtt} + \beta u_{ttt}$
$= \lambda_{tt} + (2\lambda_{ut} - \beta_{uu})u_t - \alpha_{tt}u_x + (\lambda_{uu} - 2\beta_{ut})u_t^2 - 2\alpha_{ut}u_xu_t - \beta_{uu}u_t^3 - \alpha_{uu}u_xu_t^2 +$

$(\lambda_u - 2\beta_t)u_{tt} - 2\alpha_t u_{xt} - 3\beta_u u_t u_{tt} - 2\alpha_u u_t u_{xt}$ (3.36)

$\lambda^x x = D_x^2\lambda - u_x D_x^2\alpha - u_t D_x^2\beta - 2u_x x D_x\alpha - 2u_x t D_x\beta = \lambda_{xx} + 2u_x\lambda_{ux} + u_{xx}\lambda_u + u_x^2\lambda_{uu} - u_x\alpha_{xx} - 2u_x^2\alpha_{ux} - u_x u_{xx}\alpha_u - u_x^3\alpha_{uu} - u_t\beta_{xx} - 2u_x u_t\beta_{ux} - u_{xx}u_t\beta_u - u_x^2 u_t\beta_{uu} - 2u_{xx}\alpha_x - 2u_x u_{xx}\alpha_u - 2u_{xt}\beta_x - 2u_x u_{xt}\beta_u$

$= \lambda_{xx} + u_x(2\lambda_{ux} - \alpha_{xx}) + u_{xx}(\lambda_u - 2\alpha_x) + u_x^2(\lambda_{uu} - 2\alpha_{ux}) - 3u_x u_{xx}\alpha_u - u_x^3\alpha_{uu} - u_t\beta_{xx} - 2u_x u_t\beta_{ux} - u_{xx}u_t\beta_u - u_x^2 u_t\beta_{uu} - 2u_{xt}\beta_x - 2u_x u_{xt}\beta_u$ (3.37)

$\lambda^{xxx} = D_x^3\lambda - u_x D_x^3\alpha - u_t D_x^3\beta - 3u_{xx}D_x^2\alpha - 3u_{xt}D_x^2\beta - 3u_{xxx}D_x\alpha - 3u_{xxt}D_x\beta$
$= \lambda_{xxx} + 3u_x\lambda_{uxx} + 3u_{xx}\lambda_{ux} + u_{xxx}\lambda_u + 3u_x^2\lambda_{uux} + 3u_x u_{xx}\lambda_{uu} + u_x^3\lambda_{uuu} - u_x\alpha_{xxx} - 3u_x^2\alpha_{uxx} - 3u_x u_{xx}\alpha_{ux} - u_x u_{xxx}\alpha_u - 3u_x^3\alpha_{uux} - 3u_x^2 u_{xx}\alpha_{uu} - u_x^4\alpha_{uuu} - u_t\beta_{xxx} - 3u_x u_t\beta_{uxx} - 3u_{xx}u_t\beta_{ux} - u_{xxx}u_t\beta_u - 3u_x^2 u_t\beta_{uux} - 3u_x u_{xx}u_t\beta_{uu} - u_x^3 u_t\beta_{uuu} - 3u_{xx}\alpha_{uu} - 6u_x u_{xx}\alpha_{ux} - 3u_{xx}^2\alpha_u - 3u_x^2 u_{xx}\alpha_{uu} - 3u_{xt}\beta_{xx} - 6u_x u_{xt}\beta_{ux} - 3u_{xx}u_{xt}\beta_u - 3u_x^2 u_{xt}\beta_{uu} - 3u_{xxx}\alpha_x - 3u_x u_{xxx}\alpha_u - 3u_{xxt}\beta_x - 3u_x u_{xxt}\beta_u$ (3.38)

$\lambda^{xxxx} = D_x^4\lambda - u_x D_x^4\alpha - u_t D_x^4\beta - 4u_{xx}D_x^3\alpha - 4u_{xt}D_x^3\beta - 4u_{xxx}D_x^2\alpha - 4u_{xxt}D_x^2\beta - 4u_{xxxx}D_x\alpha - 4u_{xxxt}D_x\beta$
$= \lambda_{xxxx} + u_x(4\lambda_{uxxx} - \alpha_{xxxx}) + u_{xx}(6\lambda_{uxx} - 4\alpha_{xxx}) + u_{xxx}(4\lambda_{ux} - 4\alpha_{xx}) + u_{xxxx}(\lambda_u - 4\alpha_x) + u_x^2(6\lambda_{uuxx} - 4\alpha_{uxxx}) + u_x u_{xx}(12\lambda_{uux} - 18\alpha_{uxx}) + u_{xx}^2(3\lambda_{uu} - 12\alpha_{ux}) + u_x u_{xxx}(4\lambda_{uu} - 12\alpha_{ux}) + u_x^3(4\lambda_{uuux} - 6\alpha_{uuxx}) + u_x^2 u_{xx}(6\lambda_{uuu} - 24\alpha_{uux}) + u_x^4(\lambda_{uuuu} - 4\alpha_{uuux}) - 5u_x u_{xxxx}\alpha_u - 15u_x u_{xx}^2\alpha_{uu} - 8u_x^2 u_{xxx}\alpha_{uu} - 10u_x^3 u_{xx}\alpha_{uuu} - u_x^5\alpha_{uuuu} - 8u_{xx}u_{xxx}\alpha_u - u_t\beta_{xxxx} - 4u_x u_t\beta_{uxxx} - 6u_{xx}u_t\beta_{uxx} - 4u_{xxx}u_t\beta_{ux} - u_{xxxx}u_t\beta_u - 6u_x^2 u_t\beta_{uuxx} - 12u_x u_{xx}u_t\beta_{uux} - 3u_{xx}^2 u_t\beta_{uu} - 4u_x u_{xxx}u_t\beta_{uu} - 4u_x^3 u_t\beta_{uuux} - 6u_x^2 u_{xx}u_t\beta_{uuu} - u_x^4 u_t\beta_{uuuu} - 4u_{xt}\beta_{xxx} - 12u_x u_{xt}\beta_{uxx} - 12u_{xx}u_{xt}\beta_{ux} - 4u_{xxx}u_{xt}\beta_u - 12u_x^2 u_{xt}\beta_{uux} - 12u_x u_{xx}u_{xt}\beta_{uu} - 4u_x^3 u_{xt}\beta_{uuu} - 4u_{xxt}\beta_{xx} - 8u_x u_{xxt}\beta_{ux} - 4u_{xx}u_{xxt}\beta_u - 4u_x^2 u_{xxt}\beta_{uu} - 4u_{xxxt}\beta_x - 4u_x u_{xxxt}\beta_u$ (3.39)

$\lambda^{xxxxx} = D_x^5\lambda - u_x D_x^5\alpha - u_t D_x^5\beta - 5u_{xx}D_x^4\alpha - 5u_{xt}D_x^4\beta - 5u_{xxx}D_x^3\alpha - 5u_{xxt}D_x^3\beta - 5u_{xxxx}D_x^2\alpha - 5u_{xxxt}D_x^2\beta - 5u_{xxxxx}D_x\alpha - 5u_{xxxxt}D_x\beta = \lambda_{xxxxx} + u_x(5\lambda_{uxxxx} - \alpha_{xxxxx}) + u_{xx}(10\lambda_{uxxx} - 5\alpha_{xxxx}) + u_{xxx}(10\lambda_{uxx} - 5\alpha_{xxx}) + u_{xxxx}(5\lambda_{ux} - 5\alpha_{xx}) + u_{xxxxx}(\lambda_u - 5\alpha_x) + u_x^2(10\lambda_{uuxxx} - 5\alpha_{uxxxx}) + u_x^3(10\lambda_{uuuxx} - 10\alpha_{uuxxx}) + u_x^4(5\lambda_{uuuux} - 10\alpha_{uuuxx}) + u_x^5(\lambda_{uuuuu} - 5\alpha_{uuuux}) - u_x^6\alpha_{uuuuu} + u_{xx}^2(15\lambda_{uux} - 30\alpha_{uxx}) - 15u_{xx}^3\alpha_{uu} + u_x u_{xx}(30\lambda_{uuxx} - 30\alpha_{uxxx}) + u_x u_{xxx}(20\lambda_{uux} - 25\alpha_{uxx}) + u_x u_{xxxx}(5\lambda_{uu} - 15\alpha_{ux}) - 6u_x u_{xxxxx}\alpha_u + u_{xx}u_{xxx}(10\lambda_{uu} - 35\alpha_{ux}) - 10u_{xx}u_{xxxx}\alpha_u + u_x^2 u_{xx}(30\lambda_{uuux} - 60\alpha_{uuxx}) +$

$u_x u_{xx}^2(15\lambda_{uuu} - 75\alpha_{uux}) + u_x^2 u_{xxx}(10\lambda_{uuu} - 35\alpha_{uux}) + u_x^3 u_{xx}(10\lambda_{uuuu} - 50\alpha_{uuux}) -$
$45u_x u_{xx} u_{xxx}\alpha_{uu} - 10u_x^2 u_{xxxx}\alpha_{uu} - 45u_x^2 u_{xx}^2\alpha_{uuu} - 15u_x^3 u_{xxx}\alpha_{uuu} - 15u_x^4 u_{xx}\alpha_{uuuu} -$
$5u_{xxx}^2\alpha_u - u_t\beta_{xxxxx} - 5u_x u_t\beta_{uxxxx} - 10u_{xx}u_t\beta_{uxxx} - 10u_{xxx}u_t\beta_{uxx} - 5u_{xxxx}u_t\beta_{ux} -$
$u_x xxxxu_t\beta_u - 10u_x^2 u_t\beta_{uuxxx} - 30u_x u_{xx} u_t\beta_{uuxx} - 15u_{xx}^2 u_t\beta_{uux} - 20u_x u_{xxx} u_t\beta_{uux} -$
$10u_{xx}u_{xxx}u_t\beta_{uu} - 5u_x u_{xxxx} u_t\beta_{uu} - 10u_x^3 u_t\beta_{uuuxx} - 30u_x^2 u_{xx} u_t\beta_{uuux} - 15u_x u_{xx}^2 u_t\beta_{uuu} -$
$10u_x^2 u_{xxx} u_t\beta_{uuu} - 5u_x^4 u_t\beta_{uuuux} - 10u_x^3 u_{xx} u_t\beta_{uuuu} - u_x^5 u_t\beta_{uuuuu} - 5u_x t\beta_{xxxx} - 20u_x u_{xt}\beta_{uxxx} -$
$30u_{xx}u_{xt}\beta_{uxx} - 20u_{xxx}u_{xt}\beta_{ux} - 5u_{xxxx}u_{xt}\beta_u - 30u_x^2 u_{xt}\beta_{uuxx} - 60u_x u_{xx} u_{xt}\beta_{uux} -$
$15u_{xx}^2 u_{xt}\beta_{uu} - 20u_x u_{xxx} u_{xt}\beta_{uu} - 20u_x^3 u_{xt}\beta_{uuux} - 30u_x^2 u_{xx} u_{xt}\beta_{uuu} - 5u_x^4 u_{xt}\beta_{uuuu} -$
$5u_{xxt}\beta_{xxx} - 5u_x u_{xxt}\beta_{uxx} - 15u_{xx}u_{xxt}\beta_{ux} - 5u_{xxx}u_{xxt}\beta_u - 15u_x^2 u_{xxt}\beta_{uux} - 15u_x u_{xx} u_{xxt}\beta_{uu} -$
$5u_x^3 u_{xxt}\beta_{uuu} - 5u_{xxxt}\beta_{xx} - 10u_x u_{xxxt}\beta_{ux} - 5u_{xx}u_{xxxt}\beta_u - 5u_x^2 u_{xxxt}\beta_{uu} - 5u_{xxxxt}\beta_x -$
$5u_x u_{xxxxt}\beta_u$ (3.40)

Symmetry for third order Partial Differential Equation

In order to find the symmetries of the following nonlinear third order PDE

$u_t + u_{xxx} + uu_x = 0$ (3.41)

We need to find its infinitesimal transformations, infinitesimal generators and all the groups in which it admits.

This system of equation arises in the theory of long waves in shallow water and other physical systems.

The necessary symmetry groups of transformations are of the form

$x^* = X(x,t,u;\varepsilon), t^* = T(x,t,u;\varepsilon), u^* = U(x,t,u;\varepsilon)$ (3.42)

with equivalent infinitesimals

$\alpha(x,t,u) = \frac{\partial X(x,t,u;\varepsilon)}{\partial \varepsilon}|_{\varepsilon=0}, \beta(x,t,u) = \frac{\partial X(x,t,u;\varepsilon)}{\partial \varepsilon}|_{\varepsilon=0}, \lambda(x,t,u) = \frac{\partial X(x,t,u;\varepsilon)}{\partial \varepsilon}|_{\varepsilon=0}$

We then let the generator G, of (3.41) be of the form

$G = \alpha(x,t,u)\frac{\partial}{\partial x} + \beta(x,t,u)\frac{\partial}{\partial t} + \lambda(x,t,u)\frac{\partial}{\partial u}$ (3.43)

We work out all the coefficient functions α, β, λ so that the equivalent one-parameter Lie group of transformations $x^* = X(x,t,u;\varepsilon), t^* = T(x,t,u;\varepsilon), u^* = U(x,t,u;\varepsilon)$ form a symmetry group of (3.41).

Since the equation is a third order differential equation, we use the third extension (prolongation)

$G^{[3]} = \alpha(x,t,u)\frac{\partial}{\partial x} + \beta(x,t,u)\frac{\partial}{\partial t} + \lambda(x,t,u)\frac{\partial}{\partial u} + \lambda^{t}\frac{\partial}{\partial u_t} + \lambda^{x}\frac{\partial}{\partial u_x} + \lambda^{tt}\frac{\partial}{\partial u_{tt}}\lambda^{tx}\frac{\partial}{\partial u_{tx}} +$
$\lambda^{xx}\frac{\partial}{\partial u_{xx}} + \lambda^{ttt}\frac{\partial}{\partial u_{ttt}} + \lambda^{ttx}\frac{\partial}{\partial u_{ttx}} + \lambda^{txx}\frac{\partial}{\partial u_{txx}} + \lambda^{xxx}\frac{\partial}{\partial u_{xxx}}$

When $G^{[3]}$ acts on the differential equation (3.41), we obtain

$G^{[3]}[u_t + u_{xxx} + uu_x] = 0$ (3.44)

Equation (3.44) becomes

$\alpha(x,t,u)\frac{\partial}{\partial x} + \beta(x,t,u)\frac{\partial}{\partial t} + \lambda(x,t,u)\frac{\partial}{\partial u} + \lambda^{t}\frac{\partial}{\partial u_t} + \lambda^{x}\frac{\partial}{\partial u_x} + \lambda^{tx}\frac{\partial}{\partial u_{tx}} + \lambda^{xx}\frac{\partial}{\partial u_{xx}} + \lambda^{ttt}\frac{\partial}{\partial u_{ttt}} +$
$\lambda^{ttx}\frac{\partial}{\partial u_{ttx}} + \lambda^{txx}\frac{\partial}{\partial u_{txx}} + \lambda^{xxx}\frac{\partial}{\partial u_{xxx}}][u_t + u_{xxx} + uu_x] = 0$ (3.45)

This can further be simplified to give

$\alpha(x,t,u)\frac{\partial}{\partial x}[u_t + u_{xxx} + uu_x] + \beta(x,t,u)\frac{\partial}{\partial t}[u_t + u_{xxx} + uu_x] + \lambda(x,t,u)\frac{\partial}{\partial u} + [u_t +$
$u_{xxx} + uu_x] + \lambda^{t}\frac{\partial}{\partial u_t}[u_t + u_{xxx} + uu_x] + \lambda^{x}\frac{\partial}{\partial u_x}[u_t + u_{xxx} + uu_x] + \lambda^{tx}\frac{\partial}{\partial u_{tx}}[u_t + u_{xxx} +$
$uu_x] + \lambda^{xx}\frac{\partial}{\partial u_{xx}}[u_t + u_{xxx} + uu_x] + \lambda^{ttt}\frac{\partial}{\partial u_{ttt}}[u_t + u_{xxx} + uu_x] + \lambda^{ttx}\frac{\partial}{\partial u_{ttx}}[u_t + u_{xxx} +$
$uu_x] + \lambda^{txx}\frac{\partial}{\partial u_{txx}}[u_t + u_{xxx} + uu_x] + \lambda^{xxx}\frac{\partial}{\partial u_{xxx}} = 0$ (3.46)

Then we differentiate partially with respect to the partial variables $u_t, u_x, u_{tt}, u_{tx}, u_{xx}$, $u_{ttt}, u_{ttx}, u_{txx}, u_{xxx}$ and x, t, u as algebraic variables.

Which yield the infinitesimal of the form

$\lambda u_x + \lambda^{x}u + \lambda^{t} + \lambda^{xxx}$ (3.47)

which must be satisfied ensuring that $u_t = -u_{xxx} - uu_x$ whenever it appears in the equation. When (3.34), (3.35) and (3.38) are substituted into (3.47) we obtain:

$\lambda_t - \alpha_t u_x + (\lambda_u - \beta_t)u_t - \alpha_u u_x u_t - \beta_u u_t^2 + u_x\lambda + u[\lambda_x - \beta_x u_t + (\lambda_u - \alpha_a)u_x - \alpha_u u_x^2 -$
$\beta_u u_t u_x] + \lambda_{xxx} + 3u_x\lambda_{uxx} + 3u_x^2\lambda_{uux} + 3u_{xx}\lambda_{ux} + u_x^3\lambda_{uuu} + 3u_x u_{xx}\lambda_{uu} + u_{xxx}\lambda_u -$
$3u_t(\beta_{xxx} + 3u_x\beta uxx + 3u_x^2\beta_{uux} + 3u_{xx}\beta_{ux} + 3u_x u_{xx}\beta_{uu} + u_x^3\beta_{uuu} + u_{xxx}\beta_u) - 3u_x(\alpha_{xxx} +$
$3u_x\alpha uxx + 3u_x^2\alpha_{uux} + 3u_{xx}\alpha_{ux} + 3u_x u_{xx}\alpha_{uu} + u_x^3\alpha_{uuu} + u_{xxx}\alpha_u) - 3u_{xx}(\alpha_{xx} + 2u_x\alpha ux +$
$u_{xx}\alpha_u + u_x^2\alpha_{uu}) - 3u_{xt}(\beta_{xx} + 2u_x\beta ux + u_{xx}\beta_u + u_x^2\beta_{uu}) - 3u_{xxx}(\alpha_x + u_x\alpha_u) - 3u_{xxt}(\beta_x +$
$u_x\beta_u) + \lambda_t - \alpha_t u_x(\lambda_u\beta_t)u_t - \alpha_u u_x u_t - \beta_u u_t^2 - [\lambda_{xx} + (2\lambda_{xu} - \alpha_{xx})u_x - \beta_{xx}u_t + (\lambda_{uu} -$
$2\alpha_{xu})u_x^2 - 2\beta_{xu}u_x u_t - \alpha_{uu}u_x^3 - \beta_{uu}u_x^2 u_t + (\lambda_u - 2\alpha_x)u_{xx} - 2\beta_x u_{xt} - 3\alpha_u u_x u_{xx} -$
$\beta_u u_t u_{xx} - 2\beta_u u_x u_x t] = 0$ (3.48)

When we replace u_t by $-u_{xx} - uu_x$ whenever it occurs in the equation, and equating the coefficients of the various monomials in the first, second and third order partial derivatives of u, we obtain the resulting determining equations for the infinitesimals for the equation (3.41) to be

Table 3.1: Determining Equations for Third Order PDE; Equation (3.41)

Monomials	Equations	Equation number
u_{xxt}	$\beta_x + u_x\beta_u = 0$	(i)
u_{xxx}^2	$\beta_u = 0$	(ii)
u_{xx}^2	$-3\alpha_u = 0$	(iii)
$u_x u_{xxt}$	$\beta_u = 0$	(iv)
$u_x u_{xxx}$	$\alpha_u = 0$	(v)
u_{xx}	$3\alpha_{xx} = 3\lambda_{ux}$	(vi)
$u_x u_{xx}$	$\alpha_u + 3\lambda_{uu} - 15\alpha_{ux} = 0$	(vii)
u_x^2	$3\lambda_{uux} = 0$	(viii)
u_x	$\lambda - \alpha_t + (\lambda_u - \alpha_x)u + 3\lambda_{uux} = 0$	(ix)
1	$\lambda_{xxx} + u\lambda_x\lambda_t = 0$	(x)

Results of equations (i)-(x) produce the infinitesimals α, β, λ as given below

$$\alpha = c_1 + c_3 t + c_4 x \tag{3.49a}$$

$$\beta = c_2 + 3c_4 t \tag{3.49b}$$

$$\lambda = c_3 + (-2c_4 u) \tag{3.49c}$$

We write α, β, λ in the standard basis form as follows

$$\alpha = 1.c_1 + 0.c_2 + t.c_3 + 1.c_4 x = c_1 + c_3 t + c_4 x$$

$$\beta = 0.c_1 + 1.c_2 + 0.c_3 + 3.c_4 t = c_2 + 3c_4 t$$

$$\lambda = 0.c_1 + 0.c_2 + 1.c_3 - 2.c_4.u = c_3 + (-2c_4 u) \tag{3.50}$$

We then formulate the equivalent Lie algebra of the basis generators v_1, v_2, v_3, v_4 in (3.50) of the form

$v_1 = \alpha_i \frac{\partial}{\partial x} + \beta_i \frac{\partial}{\partial t} + \lambda_i \frac{\partial}{\partial u}$: α, β, λ are the coefficients c_i in the standard solutions of α, β, λ. Thus the $v_i; i = 1, 2, 3, 4$ are obtained from the presentation in equation

(3.50) as given below

$$v_1 = \frac{\partial}{\partial x}, v_2 = \frac{\partial}{\partial t}, v_3 = \frac{\partial}{\partial u} + t\frac{\partial}{\partial x}, v_4 = x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u} \tag{3.51}$$

The Lie groups admitted by equation (3.41) are determined by solving the corresponding Lie equations which yield groups as shown below

$$v_1 = \frac{\partial}{\partial x}; G_1 : X(x,t,u;\varepsilon) \to X_1(x+\varepsilon, t, u) \tag{3.52a}$$

$$v_2 = \frac{\partial}{\partial t}; G_2 : X(x,t,u;\varepsilon) \to X_2(x, t+\varepsilon, u) \tag{3.52b}$$

$$v_3 = \frac{\partial}{\partial u} + t\frac{\partial}{\partial x}; G_3 : X(x,t,u;\varepsilon) \to X_3(x+\varepsilon t, t, u+\varepsilon) \tag{3.52c}$$

$$v_4 = x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u}; G_4 : X(x,t,u;\varepsilon) \to X_4(e^{\varepsilon}x, e^{3\varepsilon}t, e^{-2\varepsilon}u) \tag{3.52d}$$

Lie symmetry analysis for Boussinesq Equation

The Boussinesq equation is a fourth order nonlinear partial differentiation equation written as

$$\frac{\partial^2 u}{\partial t^2} = \alpha\frac{\partial^2 u}{\partial x^2} + \beta\frac{\partial^4 u}{\partial x^4} + d\frac{\partial^2}{\partial x^2}(u^2) \tag{3.53}$$

where α, β, λ non zero- real parameters.

We determine its infinitesimal transformations, infinitesimal generators and all the groups it admits. The required groups of transformations are of the form:

$$x^* = X(x,t,u;\varepsilon), t^* = T(x,t,u;\varepsilon), u^* = U(x,t,u;\varepsilon) \tag{3.54}$$

with conforming infinitesimal transformations α, β, λ where;

$\alpha(x,t,u) = \frac{\partial X(x,t,u;\varepsilon)}{\partial x}|_{\varepsilon} = 0,$

$\beta(x,t,u) = \frac{\partial T(x,t,u;\varepsilon)}{\partial t}|_{\varepsilon} = 0,$

$\lambda(x,t,u) = \frac{\partial U(x,t,u;\varepsilon)}{\partial u}|_{\varepsilon} = 0,$

The infinitesimal generator of (3.53) is given by

$$G = \alpha(x,t,u)\frac{\partial}{\partial x} + \beta(x,t,u)\frac{\partial}{\partial t} + \lambda(x,t,u)\frac{\partial}{\partial t} \tag{3.55}$$

with first, second, third and fourth extended/prolonged generators respectively as

$G^{(1)} = G + \lambda^t(x,t,u,u_t,u_x)\frac{\partial}{\partial u_t} + \lambda^x(x,t,u,u_t,u_x)\frac{\partial}{\partial u_x}$

$G^{(2)} = G^{(1)} + \lambda^{tt}(x,t,u,u_t,u_x,u_{tt},u_{tx},u_{xx})\frac{\partial}{\partial u_{tt}} + \lambda^{tx}(x,t,u,u_t,u_x,u_{tt},u_{tx},u_{xx})\frac{\partial}{\partial u_{tx}} +$
$\lambda^{xx}(x,t,u,u_t,u_x,u_{tt},u_{tx},u_{xx})\frac{\partial}{\partial u_{xx}}$

$G^{(3)} = G^{(2)} + \lambda^{ttt}\frac{\partial}{\partial u_{ttt}} + \lambda^{ttx}\frac{\partial}{\partial u_{ttx}} + \lambda^{txx}\frac{\partial}{\partial u_{txx}} + \lambda^{xxx}\frac{\partial}{\partial u_{xxx}}$

$G^{(4)} = G^{(3)} + \lambda^{tttt}\frac{\partial}{\partial u_{tttt}} + \lambda^{tttx}\frac{\partial}{\partial u_{tttx}} + \lambda^{ttxx}\frac{\partial}{\partial u_{ttxx}} + \lambda^{txxx}\frac{\partial}{\partial u_{txxx}} + \lambda^{xxxx}\frac{\partial}{\partial u_{xxxx}}$

where $\lambda^t, \lambda^x, \lambda^{tx}, \lambda^{xx}$ are known functions of the derivatives of α, β, λ and variables

$u_t, u_x, u_{tt}, u_{tx}, u_{xx}$ in which the subscripts denote partial differentiation

From (3.53), $V = (u_{tt} - \alpha u_{xx} - \beta u_{xxxx}) - 2d(u_x^2 + uu_{xx}) = 0$

By theorem 3.2, we have

$G^{(4)}V = G^{(4)}[(u_{tt} - \alpha u_{xx} - \beta u_{xxxx}) - 2d(u_x^2 + uu_{xx})] = 0$

Thus we express this to obtain

$$[\alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial t} + \lambda\frac{\partial}{\partial u} + \lambda^t\frac{\partial}{\partial u_t} + \lambda^x\frac{\partial}{\partial u_x} + \lambda^{tt}\frac{\partial}{\partial u_{tt}} + \lambda^{tx}\frac{\partial}{\partial u_{tx}} + \lambda^{xx}\frac{\partial}{\partial u_{xx}} + \lambda^{ttt}\frac{\partial}{\partial u_{ttt}} + \lambda^{ttx}\frac{\partial}{\partial u_{ttx}} + \lambda^{txx}\frac{\partial}{\partial u_{txx}} + \lambda^{xxx}\frac{\partial}{\partial u_{xxx}} + \lambda^{tttt}\frac{\partial}{\partial u_{tttt}} + \lambda^{tttx}\frac{\partial}{\partial u_{tttx}} + \lambda^{ttxx}\frac{\partial}{\partial u_{ttxx}} + \lambda^{txxx}\frac{\partial}{\partial u_{txxx}} + \lambda^{xxxx}\frac{\partial}{\partial u_{xxxx}}] \times [(u_{tt} - \alpha u_{xx} - \beta_{xxxx}) - 2\lambda(u_x^2 + uu_{xx})] = 0 \tag{3.56}$$

The infinitesimal condition (3.56) reduces to equation,

$$\lambda^{tt} - 2d\lambda u_{xx} - 4du_x\lambda^x - (\alpha + 2du)\lambda^{xx} - \beta\lambda^{xxxx} = 0 \tag{3.57}$$

with $\lambda^{tt}, \lambda^x, \lambda^{xx}, \lambda^{xxxx}$ defined explicitly as before

Substituting equations (3.35), (3.36), (3.37) and (3.39) into equation (3.57), we obtain the equation of the form

$[\lambda_{tt} + (2\lambda_{ut} - \beta_{uu})u_t - \alpha_{tt}u_x + (2\lambda_{ut} - \beta_{uu})u_t^2 - 2\alpha_{ut}u_xu_t - \beta_{uu}u_t^3 - \alpha_{uu}u_xu_t^2 + (\lambda_u - 2\beta_t)u_{tt} - 2\alpha_t u_{xt} - 3\alpha_u u_x u_{tt} - \alpha_u u_x u_{tt} - 2\alpha_u u_x u_{xt} - 2d\lambda u_{xx} - 4du_x\{_x - \beta_x t u_t + (\lambda_u - \alpha_x)u_x - \alpha_u u_x^2 - \beta_u u_t u_x\} - (\alpha + 2du)[\lambda_{xx} + (2\lambda_{xu}\alpha_{xx})u_x - \beta_{xx}u_t + (\lambda_{uu} - 2\alpha_{xu})u_x^2 - 2\beta_{xu}u_xu_t - \alpha_{uu}u_x^3 - \beta_{uu}u_x^2u_t + (\lambda_u - 2\alpha_x)u_{xx} - 2\beta_x u_{xt} - 3\alpha_u u_x u_{xx} - \beta_u u_t u_{xx} - 2\beta_u u_x u_{xt}] - \beta[-4u_{xxx}\{\alpha_x + u_x\alpha_u\} - 4u_{xxxt}\{\beta_x + u_x\beta_u\} - 4u_{xxx}\{\alpha_{xx} + 2u_x\alpha_{ux} + u_{xx}\alpha_u + u_x^2\alpha_{uu}\} - 4u_{xxt}\{\beta_{xx} + 2u_x\beta_{ux} + u_{xx}\beta_u + u_x^2\beta_{uu}\} - 4u_{xt}\{\beta_{xxx} + 3u_x\beta_{uxx} + 3u_x^2\beta_{uux} + 3u_{xx}\beta_{ux} + 3u_xu_{xx}\beta_{uu} + u_x^3\beta_{uuu} + u_{xxx}\beta_{ux}\} - 4u_{xx}(\alpha_{xxx} + 3u_x\alpha_{uxx} + 3u_x^2\alpha_{uux} + 3u_{xx}\alpha_{ux} + 3u_xu_{xx}\alpha_{uu} + u_x^3\alpha_{uuu} + u_{xxx}\alpha_u) + \{\lambda_{xxxx} + u_x\lambda_{uxxx} + 3(u_{xx}\lambda_{uxx} + u_x\lambda_{xuux} + u_x^2\lambda_{uuxx}) + 3(2u_xu_{xx}\lambda_{uux} + u_x^2\lambda_{xuux} + u_x^3\lambda_{uuux}) + (3u_x^2u_{xx}\lambda_{uuu} + u_x^3\lambda_{xuuu} + u_x^4\lambda_{uuuu}) + (u_{xxxx}\lambda_u + u_{xxx}\lambda_{xu} + u_xu_{xxx}\lambda_{uu}) + 3(u_{xxx}\lambda_{ux} + u_{xx}\lambda_{xux} + u_xu_{xx}\lambda_{uux}) + 3((u_{xx}^3 + u_xu_{xx})\lambda_{uu} + u_xu_{xx}\lambda_{xuu} + u_x^2u_{xx}\lambda uuu)\} - u_t\{\beta_{xxxx} + u_x\beta_{uxxx} + 3(u_{xx}\beta_{uxx} + u_x\beta_{xuux} + u_x^2\beta_{uuxx}) + 3(2u_xu_{xx}\beta_{xuux} + u_x^3\beta_{uuux}) + (3u_x^2u_{xx}\beta_{uuu} + u_x^3\beta_{xuuu} + u_x^4\beta_{uuuu}) + (u_{xxxx}\beta_u + u_{xxx}\beta_{xu} + u_xu_{xxx}\beta_{uu}) + 3(u_{xxx}\beta_{ux} + u_{xx}\beta_{xux} + u_xu_{xx}\beta_{uuuu} + 3((u_{xx}^3 + u_xu_{xx}\beta_{uu} + u_xu_{xx}\beta_{xuu} + u_x^2u_{xx}\beta_{uuu})\} - u_x\{\alpha_{xxxx} + u_x\alpha_{uxxx} + 3(u_{xx}\alpha_{uxx} + u_x\alpha_{xuux} + u_x^2\alpha_{uuxx}) + 3(2u_xu_{xx}\alpha_{xuux} + u_x^3\alpha_{uuux}) + (3u_x^2u_{xx}\alpha_{uuu} + u_x^3\alpha_{xuuu} + u_x^4\alpha_{uuuu}) + (u_{xxxx}\alpha_u + u_{xxx}\alpha_{xu} + u_xu_{xxx}\alpha_{uu}) + 3(u_{xxx}\alpha_{ux} + u_{xx}\alpha_{xux} + u_xu_{xx}\alpha_{uuuu} + 3((u_{xx}^3 + u_xu_{xx}\alpha_{uu} + u_xu_{xx}\alpha_{xuu} +$

$u_x^2u_{xx}\alpha_{uuu})\} - 4u_{xt}(\beta_{xxx} + 3u_x\beta_{uxx} + 3u_x^2\beta_{uux} + 3u_{xx}\beta_{ux} + 3u_xu_{xx}\beta_{uu} + u_x^3\beta_{uuu} + u_{xxx}\beta_u) - 4u_{xx}(\alpha_{xxx} + 3u_x\alpha_{uxx} + 3u_{xuux}^2 + 3u_{xx}\alpha_{ux} + 3u_xu_{xx}\alpha_{uu} + u_x^3\alpha_{uuu} + u_{xxx}\alpha_u)]] = 0$ (3.58)

whenever $(u_{tt} - \alpha u_{xx} - \beta u_{xxxx}) = 2d(u_x^2 + uu_{xx}$

When we equate to zero the coefficients of the monomial terms, we obtain equations in the partial derivatives of infinitesimals α, β, λ which yield:

$\alpha = m_2 + m_3x$ (3.59)

$\beta = m_1 + 2m_3t$ (3.60)

$\lambda = m_3[x + 2t - (\frac{\alpha}{d} - 2u)]$ (3.61)

The infinitesimal generators v_i are obtained to be

$v_1 = \frac{\partial}{\partial t}, v_2 = \frac{\partial}{\partial x}, v_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - [\frac{\alpha}{d} - 2u]2t\frac{\partial}{\partial u}$ (3.62)

The terms $\lambda^{tt}, \lambda^x, \lambda^{xx}, \lambda^{xxxx}$ in the prolongation of the generator are expressed as functions of $\alpha, \beta, \lambda, u$.

The one-parameter groups G_i admitted by the infinitesimal generators, v_i are determined by solving the corresponding Lie equations which yield groups as follows:

$v_1 = \frac{\partial}{\partial t}; G_1 : X(x, t, u; \varepsilon) \to X_1(x, t + \varepsilon, u)$ (3.63)

$v_2 = \frac{\partial}{\partial x}; G_2 : X(x, t, u; \varepsilon) \to X_2(x, t + \varepsilon, u)$ (3.64)

$v_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - [\frac{\alpha}{d} - 2u]\frac{\partial}{\partial u}; G_3 : X(x, t, u; \varepsilon) \to X_3(e^{\varepsilon}x, e^{2\varepsilon}t, (e^{2\varepsilon} - \frac{\alpha}{d}a)u)$ (3.65)

where a is arbitrary solution of the fourth order nonlinear Boussinesq equation.

These groups above are all trivial groups given as:

$G_1 : X(x, t, u; \varepsilon) \to X_1(x, t + \varepsilon, u)$

$G_2 : X(x, t, u; \varepsilon) \to X_2(x, t + \varepsilon, u)$

$G_1 : X(x, t, u; \varepsilon) \to X_3(e^{\varepsilon}x, e^{2\varepsilon}t, (e^{2\varepsilon} - \frac{\alpha}{d}a)u)$

Lie symmetry analysis of the Wave Equation

The wave equation expressed in two dimensions is of the form

$\frac{\partial^2u}{\partial t^2} - \frac{\partial^2u}{\partial x^2} - \frac{\partial^2u}{\partial y^2} = 0$ (3.66)

We determine its infinitesimal transformations, infinitesimal generators and all the groups it admits.

We let the infinitesimal generator G for (3.66), be of the form

$$G = \alpha(x,t,y,u)\frac{\partial}{\partial x} + \beta(x,t,y,u)\frac{\partial}{\partial t} + \mu(x,t,y,u)\frac{\partial}{\partial y} + \lambda(x,t,y,u)\frac{\partial}{\partial u} \quad (3.67)$$

Then we determine infinitesimals $\alpha, \beta, \mu, \lambda$ so that the corresponding one-parameter Lie group of transformations,

$x^* = X(x,t,y,u;\varepsilon), t^* = T(x,t,y,u;\varepsilon), y^* = T(x,t,y,u;\varepsilon), u^* = U(x,t,y,u;\varepsilon)$

form a symmetry group of (3.66).

From theorem 3.2, we know that the equation

$$G^{(2)}[\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}] = 0 \quad (3.68)$$

is the symmetry condition for (3.66) and we note that $G^{(2)}$ is the second prolongation with

$G^{(2)} = \alpha(x,t,y,u)\frac{\partial}{\partial x} + \beta(x,t,y,u)\frac{\partial}{\partial t} + \mu(x,t,y,u)\frac{\partial}{\partial y} + \lambda(x,t,y,u)\frac{\partial}{\partial u} + \lambda^t\frac{\partial}{\partial u_t} + \lambda^x\frac{\partial}{\partial u_x} + \lambda^y\frac{\partial}{\partial u_y} + \lambda^{xx}\frac{\partial}{\partial u_{xx}} + \lambda^{yy}\frac{\partial}{\partial u_{yy}} + \lambda^{xt}\frac{\partial}{\partial u_{xt}} + \lambda^{xy}\frac{\partial}{\partial u_{xy}} + \lambda^{xt}\frac{\partial}{\partial u_{xt}} + \lambda^{yt}\frac{\partial}{\partial u_{yt}} + \lambda^{tt}\frac{\partial}{\partial u_{tt}}$

Hence equation (3.68) becomes

$[\alpha(x,t,y,u)\frac{\partial}{\partial x} + \beta(x,t,y,u)\frac{\partial}{\partial t} + \mu(x,t,y,u)\frac{\partial}{\partial y} + \lambda(x,t,y,u)\frac{\partial}{\partial u} + \lambda^t\frac{\partial}{\partial u_t} + \lambda^x\frac{\partial}{\partial u_x} + \lambda^y\frac{\partial}{\partial u_y} + \lambda^{xx}\frac{\partial}{\partial u_{xx}} + \lambda^{yy}\frac{\partial}{\partial u_{yy}} + \lambda^{xt}\frac{\partial}{\partial u_{xt}} + \lambda^{xy}\frac{\partial}{\partial u_{xy}} + \lambda^{xt}\frac{\partial}{\partial u_{xt}} + \lambda^{yt}\frac{\partial}{\partial u_{yt}} + \lambda^{tt}\frac{\partial}{\partial u_{tt}}][\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}] = 0$

Upon expansion it takes the form;

$$[\alpha(x,t,y,u)\frac{\partial}{\partial x}[\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}] + \beta(x,t,y,u)\frac{\partial}{\partial t}[\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}] + \mu(x,t,y,u)\frac{\partial}{\partial y}[\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}] + \lambda(x,t,y,u)\frac{\partial}{\partial u}[\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}] + \lambda^t\frac{\partial}{\partial u_t}[\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}] + \lambda^x\frac{\partial}{\partial u_x} + [\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}]\lambda^y\frac{\partial}{\partial u_y}[\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}] + \lambda^{xx}\frac{\partial}{\partial u_{xx}}[\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}] + \lambda^{yy}\frac{\partial}{\partial u_{yy}}[\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}] + \lambda^{xt}\frac{\partial}{\partial u_{xt}}[\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}] + \lambda^{xy}\frac{\partial}{\partial u_{xy}}[\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}] + \lambda^{xt}\frac{\partial}{\partial u_{xt}}[\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}] + \lambda^{yt}\frac{\partial}{\partial u_{yt}}[\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}] + \lambda^{tt}\frac{\partial}{\partial u_{tt}}[\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}] = 0 \quad (3.69)$$

Thus we obtain the infinitesimals condition to be

$$\lambda^{tt} - \lambda^{xx} - \lambda^{yy} = 0 \quad (3.70)$$

which must be fulfilled whenever $u_{tt} = u_{xx} + u_{yy}$.

When (3.36) and (3.37)are substituted into (3.70)

we obtain:

$\lambda_{tt} + 2u_t\lambda_{ut} + u_{tt}\lambda_u + u_t^2\lambda_{uu} - u_x(\alpha_{tt} + 2u_t\alpha_{ut} + u_{tt}\alpha_u + u_t^2\alpha_{uu}) - u_y(\mu_{tt} + 2u_t\mu_{ut} + u_{tt}\mu_u + u_t^2\mu_{uu}) - u_x(\beta_{tt} + 2u_t\beta_{ut} + u_{tt}\beta_u + u_t^2\beta_{uu}) - 2u_{xt}(\alpha_t + u_t\alpha_u) - 2u_{ty}(\mu_t + u_t\mu_u) -$

$2u_{ty}(\beta_t+u_t\beta_u)-[\lambda_{xx}+2u_x\lambda_{ux}+u_{xx}\lambda_u+u_x^2\lambda_{uu}-u_x(\alpha_{xx}+2u_x\alpha_{ux}+u_{xx}\alpha_u+u_x^2\alpha_{uu}-u_x)-u_y(\mu_{xx}+2u_x\mu_{ux}+u_{xx}\mu_u+u_x^2\mu_{uu}-u_x)-u_t(\beta_{xx}+2u_x\beta_{ux}+u_{xx}\beta_u+u_x^2\beta_{uu}-u_x)-2u_{xx}(\alpha_x+u_x\alpha_u)-2u_{yx}(\mu_x+u_x\mu_u)-2u_{tx}(\beta_x+u_x\beta_u)]-[\lambda_{yy}+2u_y\lambda_{uy}+u_{yy}\lambda_u+u_y^2\lambda_{uu}-u_y(\alpha_{yy}+2u_y\alpha_{uy}+u_{yy}\alpha_u+u_y^2\alpha_{uu}-u_y)-u_y(\mu_{yy}+2u_y\mu_{uy}+u_{yy}\mu_u+u_y^2\mu_{uu}-u_y)-u_t(\beta_{yy}+2u_y\beta_{uy}+u_{yy}\beta_u+u_y^2\beta_{uu}-u_y)-2u_{yy}(\alpha_y+u_y\alpha_u)-2u_{yx}(\mu_y+u_y\mu_u)-2u_{ty}(\beta_y+u_y\beta_u)]=0$ (3.71)

On replacing u_{tt} by $u_{xx}+u_{yy}$ wherever it occurs in the equation and equating the coefficients of the various monomials in the first and second order partial derivatives of u,we obtain the resulting equations for the Wave equation (3.66) as tabulated below i.e.

Table 3.2: Determining Equations for the Wave Equation; equation(3.66)

Monomials	Equations	Equation number
u_{xxt}	$\beta_x+u_x\beta_u=0$	(i)
u_{xxx}^2	$\beta_u=0$	(ii)
u_{xx}^2	$-3\alpha_u=0$	(iii)
u_xu_{xxt}	$\beta_u=0$	(iv)
u_xu_{xxx}	$\alpha_u=0$	(v)
u_{xx}	$3\alpha_{xx}=3\lambda_{ux}$	(vi)
u_xu_{xx}	$3\lambda_{uu}-6\alpha_{ux}-9\alpha_{ux}=0$	(vii)
u_x^2	$3\lambda_{uux}=0$	(viii)
u_x	$\lambda-\alpha_t+(\lambda_u-\alpha_x)u+3\lambda_{uux}=0$	(ix)
1	$\lambda_{xxx}+u\lambda_x\lambda_t=0$	(x)

The solutions of (i)-(x) yield the infinitesimals α,β,μ,λ as below, [6].

$\alpha=c_1+c_4x-c_5y+c_6t+c_8(x^2-y^2+t^2)+2c_9xy+2c_{10}xt$ (3.72a)

$\beta=c_3+c_6x-c_7y+c_4t+c_10(x^2-y^2+t^2)+2c_9ty+2c_8xt$ (3.72b)

$\mu = 1.c_2 + 1c_4y - 1c_5x + 1.c_7t + 2.c_8xy + c_9.(-x^2 + y^2 + t^2) + 2.c_{10}yt$ (3.72c)

$\lambda = (c_{11} - c_8x - c_9y - c_{10}t)u + \alpha(x, y, t)$ (3.72d)

α is an arbitrary solution of the wave equation.

We express $\alpha, \beta, \mu, \lambda$ in the standard basis form:

$\alpha = 1.c_1 + 0.c_2 + 0.c_3 + c_4x - c_5y + c_6t + 0.c_7 + c_8(x^2 - y^2 + t^2) + 2c_9xy + 2c_{10}xt + 0.c_{11} + 0.c_\alpha$

$\mu = 0.c_1 + 1.c_2 + 0.c_3 + 1c_4y + 1c_5x + 0.c_6 + 1.c_7t + 2.c_8xy + c_9.(-x^2 + y^2 + t^2) + 2.c_{10}yt + 0.c_{11} + 0.c_\alpha$

$\beta = 0.c_1 + 0.c_2 + 1.c_3 + 1.c_4t + 0.c_5 + 1.c_6.x + 1.c_7y + 2.c_8xt + 2.c_9.ty + c_{10}.1(x^2 - y^2 + t^2) + 0.c_{11} + 0.c_\alpha$

$\lambda = 0.c_1 + 0.c_2 + 0.c_3 + 0.c_4 + 0.c_5 + 0.c_6 + 0.c_7 - 1.c_8.1.x.u - c_9.1.y.u - c_{10}.1.u.t + 1.c_{11}.u + 1.c_\alpha\alpha$

We form the corresponding Lie Algebra of the basis generators $v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_\alpha$ of the form

$v_i = \alpha_i\frac{\partial}{\partial x} + \mu_i\frac{\partial}{\partial x} + \beta_i\frac{\partial}{\partial x} + \lambda_i\frac{\partial}{\partial x} : \alpha_i, \mu_i, \beta_i, \lambda_i$ are the coefficients c_i in the standard solutions of $\alpha, \beta, \mu, \lambda$.

Hence the $v_i's$ are obtained from the tabulation as follows:

$v_1 = \frac{\partial}{\partial x}, v_2 = \frac{\partial}{\partial y}, v_3 = \frac{\partial}{\partial t}, v_4 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + t\frac{\partial}{\partial t}, v_5 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + t\frac{\partial}{\partial t}, v_6 = t\frac{\partial}{\partial x} + x\frac{\partial}{\partial t}, v_7 = t\frac{\partial}{\partial y} + y\frac{\partial}{\partial t}, v_8 = (x^2 - y^2 + t^2)\frac{\partial}{\partial x} + 2yx\frac{\partial}{\partial y} + 2xt\frac{\partial}{\partial t} - xu\frac{\partial}{\partial u}, v_9 = 2xy\frac{\partial}{\partial x}(-x^2 + y^2 + t^2)\frac{\partial}{\partial y} + 2yt\frac{\partial}{\partial t} - yu\frac{\partial}{\partial u}, v_{10} = 2xt\frac{\partial}{\partial x} + 2yt\frac{\partial}{\partial y}(x^2 + y^2 + t^2)\frac{\partial}{\partial y} - tu\frac{\partial}{\partial u}, v_{11} = u\frac{\partial}{\partial u}, v_\alpha = \alpha(x, y, t)\frac{\partial}{\partial u}v_\alpha = \alpha(x, y, t)$ (3.73)

To determine the one-parameter groups G_i admitted by equation (3.66) from the infinitesimal generators, $v_i's$, we solve the corresponding Lie equations which give the groups as shown below. Olver[28]

$v_1 = \frac{\partial}{\partial x}; G_1 : X(x, t, y, u : \varepsilon) \to X_1(x+\varepsilon, y, t, u)$ (3.74a)

$v_2 = \frac{\partial}{\partial y}; G_2 : X(x, t, y, u : \varepsilon) \to X_2(x, y+\varepsilon, t, u)$ (3.74b)

$v_3 = \frac{\partial}{\partial t}; G_3 : X(x, t, y, u : \varepsilon) \to X_3(x, y, t+\varepsilon, u)$ (3.74c)

$v_4 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + t\frac{\partial}{\partial t}; G_4 : X(x, t, y, u : \varepsilon) \to X_4(e^\varepsilon x, e^\varepsilon y, e^\varepsilon t, u)$ (3.74d)

$v_5 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + t\frac{\partial}{\partial t}; G_5 : X(x, t, y, u : \varepsilon) \to X_5(x - \varepsilon, y, y + \varepsilon x, e^\varepsilon t, u)$ (3.74e)

$$v_6 = t\frac{\partial}{\partial x}+x\frac{\partial}{\partial t}; G_6 : X(x,t,y,u;\varepsilon) \to X_6(x+\varepsilon t, y, t+\varepsilon x, u) \qquad (3.74f)$$

$$v_7 = t\frac{\partial}{\partial x}+y\frac{\partial}{\partial t}; G_7 : X(x,t,y,u;\varepsilon) \to X_7(x, y+\varepsilon t, t+\varepsilon y, u) \qquad (3.74g)$$

$$v_8 = (x^2 - y^2 + t^2)\frac{\partial}{\partial x} + 2yx\frac{\partial}{\partial y} + 2xt\frac{\partial}{\partial t} - xu\frac{\partial}{\partial u}; G_8 : X(x,t,y,u;\varepsilon) \to$$
$$X_8(\tfrac{x+\varepsilon(t^2-x^2-y^2)}{1-2\varepsilon x-\varepsilon^2(t^2-x^2-y^2)}, \tfrac{y}{1-2\varepsilon x-\varepsilon^2(t^2-x^2-y^2)}, \tfrac{t}{1-2\varepsilon x-\varepsilon^2(t^2-x^2-y^2)},$$
$$u\sqrt{1-2\varepsilon x-\varepsilon^2(t^2-x^2-y^2)}) \qquad (3.74h)$$

$$v_9 = 2xy\frac{\partial}{\partial x} + (-x^2 + y^2 + t^2)\frac{\partial}{\partial y} + 2yt\frac{\partial}{\partial t} - yu\frac{\partial}{\partial u}; G_9 : X(x,t,y,u;\varepsilon) \to$$
$$X_9(\tfrac{x}{1-2\varepsilon y-\varepsilon^2(t^2-x^2-y^2)}, \tfrac{y+\varepsilon(t^2-x^2-y^2)}{1-2\varepsilon y-\varepsilon^2(t^2-x^2-y^2)}, \tfrac{t}{1-2\varepsilon y-\varepsilon^2(t^2-x^2-y^2)},$$
$$u\sqrt{1-2\varepsilon y-\varepsilon^2(t^2-x^2-y^2)}) \qquad (3.74i)$$

$$v_{10} = 2xt\frac{\partial}{\partial x} + 2yt\frac{\partial}{\partial y} + (x^2 + y^2 + t^2)\frac{\partial}{\partial t} - tu\frac{\partial}{\partial u}; G_{10} : X(x,t,y,u;\varepsilon) \to$$
$$X_{10}(\tfrac{x}{1-2\varepsilon t-\varepsilon^2(t^2-x^2-y^2)}, \tfrac{y}{1-2\varepsilon t-\varepsilon^2(t^2-x^2-y^2)}, \tfrac{t+\varepsilon(t^2-x^2-y^2)}{1-2\varepsilon t-\varepsilon^2(t^2-x^2-y^2)},$$
$$u\sqrt{1-2\varepsilon t-\varepsilon^2(t^2-x^2-y^2)}) \qquad (3.74j)$$

$$v_{11} = u\frac{\partial}{\partial u}; G_{11} : X(x,y,t,u:\varepsilon) \to X_{11}(x,y,t,e^{\varepsilon},u) \qquad (3.74k)$$

3.6 Canonical Variables

This technique was applied in integrating first order equation with a known infinitesimal symmetry.

This method is of great importance since it was used in eliminating the explicit dependence of equation on one of the variables either x or t thus integrating the equation by quadrature.

It was used in the reduction of higher order equations.

CHAPTER FOUR

LIE SYMMETRY SOLUTIONS OF SAWADA-KOTERA EQUATION

4.1 Introduction

In this chapter, Sawada-Kotera equation is solved using Lie symmetry analysis.

4.2 Infinitesimal Transformations

The solution of Sawada-Kotera equation of the form

$$u_t+45u^2u_x+15u_xu_{xx}+15uu_{xxx}+u_{xxxxx}=0 \quad (4.1)$$

can be obtained analytically. In this study, we have solved Sawada-Kotera equation analytically using Lie symmetry analysis technique.

We generated infinitesimal generators, infinitesimal transformations and the groups which the equation admits.

The groups of transformation required were of the form;

$$x^* = X(x,t,u;\epsilon) \quad (4.2a)$$

$$t^* = T(x,t,u;\epsilon) \quad (4.2b)$$

$$u^* = U(x,t,u;\epsilon) \quad (4.2c)$$

And their corresponding infinitesimal transformations α, β, in which

$$\alpha(x,t,u) = \frac{\partial X(x,t,u;\epsilon)}{\partial x}|_{\epsilon=0} \quad (4.3a)$$

$$\beta(x,t,u) = \frac{\partial T(x,t,u;\epsilon)}{\partial t}|_{\epsilon=0} \quad (4.3b)$$

$$\lambda(x,t,u) = \frac{\partial U(x,t,u;\epsilon)}{\partial u}|_{\epsilon=0} \quad (4.3c)$$

4.3 Infinitesimal Generator and Prolongations

The infinitesimal generator of equation (4.1) is

$$G = \alpha(x,t,u)\frac{\partial}{\partial x} + \beta(x,t,u)\frac{\partial}{\partial t} + \lambda(x,t,u)\frac{\partial}{\partial u} \quad (4.4)$$

with $u = u(x,t)$[6]

Since the equation is a fifth order differential equation, we used the fifth extension (prolongation) of the generator.

The prolongations of the generator from the first to the fifth are:[28]

$$G^{[1]} = G^{[0]} + \lambda^t\frac{\partial}{\partial u_t} + \lambda^x\frac{\partial}{\partial u_x} \quad (4.5)$$

$$G^{[2]} = G^{[1]} + \lambda^{tt}\frac{\partial}{\partial u_{tt}} + \lambda^{tx}\frac{\partial}{\partial u_{tx}} + \lambda^{xx}\frac{\partial}{\partial u_{xx}} \quad (4.6)$$

$$G^{[3]} = G^{[2]} + \lambda^{ttt}\frac{\partial}{\partial u_{ttt}} + \lambda^{ttx}\frac{\partial}{\partial u_{ttx}} + \lambda^{txx}\frac{\partial}{\partial u_{txx}} + \lambda^{xxx}\frac{\partial}{\partial u_{xxx}} \quad (4.7)$$

$$G^{[4]} = G^{[3]} + \lambda^{tttt}\frac{\partial}{\partial u_{tttt}} + \lambda^{tttx}\frac{\partial}{\partial u_{tttu}} + \lambda^{ttxx}\frac{\partial}{\partial u_{ttxx}} + \lambda^{txxx}\frac{\partial}{\partial u_{txxx}} + \lambda^{xxxx}\frac{\partial}{\partial u_{xxxx}} \quad (4.8)$$

$$G^{[5]} = G^{[4]} + \lambda^{ttttt}\frac{\partial}{\partial u_{ttttt}} + \lambda^{ttttx}\frac{\partial}{\partial u_{ttttx}} + \lambda^{tttxx}\frac{\partial}{\partial u_{tttxx}} + \lambda^{ttxxx}\frac{\partial}{\partial u_{ttxxx}} + \lambda^{txxxx}\frac{\partial}{\partial u_{txxxx}} + \lambda^{xxxxx}\frac{\partial}{\partial u_{xxxxx}} \quad (4.9)$$

By theorem 3.2, the fifth prolongation acts on equation (4.1)

$$G^{[5]}[u_t + 45u^2u_x + 15u_xu_{xx} + 15uu_{xxx} + u_{xxxxx}] = 0$$

$$\alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial t} + \lambda\frac{\partial}{\partial u} + \lambda^t\frac{\partial}{\partial u_t} + \lambda^x\frac{\partial}{\partial u_x} + \lambda^{tt}\frac{\partial}{\partial u_{tt}} + \lambda^{tx}\frac{\partial}{\partial u_{tx}} + \lambda^{xx}\frac{\partial}{\partial u_{xx}} + \lambda^{ttt}\frac{\partial}{\partial u_{ttt}} + \lambda^{ttx}\frac{\partial}{\partial u_{ttx}} + \lambda^{txx}\frac{\partial}{\partial u_{txx}} + \lambda^{xxx}\frac{\partial}{\partial u_{xxx}} + \lambda^{tttt}\frac{\partial}{\partial u_{tttt}} + \lambda^{tttx}\frac{\partial}{\partial u_{tttu}} + \lambda^{ttxx}\frac{\partial}{\partial u_{ttxx}} + \lambda^{txxx}\frac{\partial}{\partial u_{txxx}} + \lambda^{xxxx}\frac{\partial}{\partial u_{xxxx}} + \lambda^{ttttt}\frac{\partial}{\partial u_{ttttt}} + \lambda^{ttttx}\frac{\partial}{\partial u_{ttttx}} + \lambda^{tttxx}\frac{\partial}{\partial u_{tttxx}} + \lambda^{ttxxx}\frac{\partial}{\partial u_{ttxxx}} + \lambda^{txxxx}\frac{\partial}{\partial u_{txxxx}} + \lambda^{xxxxx}\frac{\partial}{\partial u_{xxxxx}}] \times [u_t + 45u^2u_x + 15u_xu_{xx} + 15uu_{xxx} + u_{xxxxx}] = 0 \quad (4.10)$$

Thus we presented $\alpha\frac{\partial}{\partial x}[u_t + 45u^2u_x + 15u_xu_{xx} + 15uu_{xxx} + u_{xxxxx}] + \beta\frac{\partial}{\partial t}[u_t + 45u^2u_x + 15u_xu_{xx} + 15uu_{xxx} + u_{xxxxx}] + \lambda\frac{\partial}{\partial u}[u_t + 45u^2u_x + 15u_xu_{xx} + 15uu_{xxx} + u_{xxxxx}] + \lambda^t\frac{\partial}{\partial u_t}[u_t + 45u^2u_x + 15u_xu_{xx} + 15uu_{xxx} + u_{xxxxx}] + \lambda^x\frac{\partial}{\partial u_x}[u_t + 45u^2u_x + 15u_xu_{xx} + 15uu_{xxx} + u_{xxxxx}] + \lambda^{tt}\frac{\partial}{\partial u_{tt}}[u_t + 45u^2u_x + 15u_xu_{xx} + 15uu_{xxx} + u_{xxxxx}] + \lambda^{tx}\frac{\partial}{\partial u_{tx}}[u_t + 45u^2u_x + 15u_xu_{xx} + 15uu_{xxx} + u_{xxxxx}] + \lambda^{xx}\frac{\partial}{\partial u_{xx}}[u_t + 45u^2u_x + 15u_xu_{xx} + 15uu_{xxx} + u_{xxxxx}] + \lambda^{ttt}\frac{\partial}{\partial u_{ttt}}[u_t + 45u^2u_x + 15u_xu_{xx} + 15uu_{xxx} + u_{xxxxx}] + \lambda^{ttx}\frac{\partial}{\partial u_{ttx}}[u_t + 45u^2u_x + 15u_xu_{xx} + 15uu_{xxx} + u_{xxxxx}] + \lambda^{txx}\frac{\partial}{\partial u_{txx}}[u_t + 45u^2u_x + 15u_xu_{xx} + 15uu_{xxx} + u_{xxxxx}] + \lambda^{xxx}\frac{\partial}{\partial u_{xxx}}[u_t + 45u^2u_x + 15u_xu_{xx} + 15uu_{xxx} + u_{xxxxx}] + \lambda^{tttt}\frac{\partial}{\partial u_{tttt}}[u_t + 45u^2u_x + 15u_xu_{xx} + 15uu_{xxx} + u_{xxxxx}] + \lambda^{tttx}\frac{\partial}{\partial u_{tttu}}[u_t + 45u^2u_x + 15u_xu_{xx} + 15uu_{xxx} + u_{xxxxx}] +$

$\lambda^{ttxx}\frac{\partial}{\partial u_{ttxx}}[u_t + 45u^2u_x + 15u_xu_{xx} + 15uu_{xxx} + u_{xxxxx}] + \lambda^{txxx}\frac{\partial}{\partial u_{txxx}}[u_t + 45u^2u_x + 15u_xu_{xx}+15uu_{xxx}+u_{xxxxx}]+\lambda^{xxxx}\frac{\partial}{\partial u_{xxxx}}[u_t+45u^2u_x+15u_xu_{xx}+15uu_{xxx}+u_{xxxxx}]+ \lambda^{ttttt}\frac{\partial}{\partial u_{ttttt}}[u_t + 45u^2u_x + 15u_xu_{xx} + 15uu_{xxx} + u_{xxxxx}] + \lambda^{ttttx}\frac{\partial}{\partial u_{ttttx}}[u_t + 45u^2u_x + 15u_xu_{xx}+15uu_{xxx}+u_{xxxxx}]+\lambda^{tttxx}\frac{\partial}{\partial u_{tttxx}}[u_t+45u^2u_x+15u_xu_{xx}+15uu_{xxx}+u_{xxxxx}]+ \lambda^{ttxxx}\frac{\partial}{\partial u_{ttxxx}}[u_t+45u^2u_x+15u_xu_{xx}+15uu_{xxx}+u_{xxxxx}]+\lambda^{txxxx}\frac{\partial}{\partial u_{txxxx}}[u_t+45u^2u_x+ 15u_xu_{xx} + 15uu_{xxx} + u_{xxxxx}] + \lambda^{xxxxx}\frac{\partial}{\partial u_{xxxxx}}[u_t + 45u^2u_x + 15u_xu_{xx} + 15uu_{xxx} + u_{xxxxx}] = 0$ (4.11)

Therefore, we obtained

$\alpha u_{xt} + 90\alpha uu_x^2 + 45\alpha u^2u_{xx} + 15\alpha u_{xx}^2 + 30\alpha u_xu_{xxx} + 15\alpha uu_{xxx} + \alpha u_{xxxxxx} + \beta u_{tt} + 90\beta uu_xu_t+45\beta u^2u_{xt}+15\beta u_{xt}u_{xx}+15\beta u_xu_{xxt}+15\beta u_tu_{xxx}+15\beta uu_{xxxt}+\beta u_{xxxxxt}+ 90\lambda uu_x + 15\lambda u_{xxx} + \lambda^t + 45\lambda^xu^2 + 15\lambda^xu_{xx} + 15\lambda^{xx}u_x + 15\lambda^{xxx}u + \lambda^{xxxxx} = 0$ (4.12)

From equation (4.1), we know that

$$u_{xxxxx} = -u_t-45u^2u_x-15u_xu_{xx}-15uu_{xxx} \quad (4.13)$$

and

$$\begin{aligned} u_{xxxxxx} &= (u_{xxxxx})' \\ &= (-u_t - 45u^2u_x - 15u_xu_{xx} - 15uu_{xxx})' \\ &= -u_{xt}-90uu_x^2-45u^2u_{xx}-15u_{xx}^2-30u_xu_{xxx}-15uu_{xxxx} \end{aligned} \quad (4.14)$$

Also from equation (4.1) we have

$$u_t = -45u^2u_x - 15u_xu_{xx} - 15uu_{xxx} - u_{xxxxx} \quad (4.15)$$

and

$$\begin{aligned} u_{tt} &= (u_t)' \\ &= (-45u^2u_x - 15u_xu_{xx} - 15uu_{xxx} - u_{xxxxx})' \\ &= -90uu_xu_t-45u^2u_{xt}-15u_{xt}u_{xx}-15u_xu_{xxt}-15u_tu_{xxx}-15uu_{xxxt}-u_{xxxxxt} \end{aligned} \quad (4.16)$$

Substituting equations (4.14) and (4.16) into equation (4.12), we obtained

$\alpha u_{xt} + 90\alpha uu_x^2 + 45\alpha u^2u_{xx} + 15\alpha u_{xx}^2 + 30\alpha u_xu_{xxx} + 15\alpha uu_{xxxx} - \alpha u_{xt} - 90\alpha uu_x^2 - 45\alpha u^2u_{xx} - 15\alpha u_{xx}^2 - 30\alpha u_xu_{xxx} - 15\alpha uu_{xxxx} - 90\beta uu_xu_t - 45\beta u^2u_{xt} - 15\beta u_{xt}u_{xx} - 15\beta u_xu_{xxt} - 15_tu_{xxx} - 15\beta uu_{xxxt} - \beta u_{xxxxxt} + 90\beta uu_xu_t + 45\beta u^2u_{xt} + 15\beta u_{xt}u_{xx} + 15\beta u_xu_{xxt} + 15\beta u_tu_{xxx} + 15\beta uu_{xxxt} + \beta u_{xxxxxt} + 90\lambda uu_x + 15\lambda u_{xxx} + \lambda^t + 45\lambda^xu^2 +$

$15\lambda^{x}u_{xx} + 15\lambda^{xx}u_{x} + 15\lambda^{xxx}u + \lambda^{xxxxx} = 0$ (4.17)

On simplifying, we had

$90\lambda uu_{x} + 15\lambda u_{xxx} + \lambda^{t} + 45\lambda^{x}u^{2} + 15\lambda^{x}u_{xx} + 15\lambda^{xx}u_{x} + 15\lambda^{xxx}u + \lambda^{xxxxx} = 0$ (4.18)

We replaced the generated coefficients in equations (3.34), (3.35), (3.37), (3.38) and (3.40) into equation (4.18) and ensuring that we replaced $u_t = -45u^2u_x - 15u_xu_{xx} - 15uu_{xxx} - u_{xxxxx}$ whenever it appeared in the equation to obtain

$90\lambda uu_x + 15\lambda u_{xxx} + \lambda_t - 45u^2u_x\lambda_u - 15u_xu_{xx}\lambda_u - 15uu_{xxx}\lambda_u - u_{xxxxx}\lambda_u + 45u^2u_x\beta_t + 15u_xu_{xx}\beta_t + 15uu_{xxx}\beta_t + u_{xxxxx}\beta_t - u_x\alpha_t + 45u^2u_x^2\alpha u + 15u_x^2u_{xx}\alpha_u + 15uu_xu_{xxx}\alpha_u + u_xu_{xxxxx}\alpha_u - 2025u^4u_x^2\beta_u - 1350u^2u_x^2u_{xx}\beta_u - 1350u^3u_xu_{xxx}\beta_u - 90u^2u_xu_{xxxxx}\beta_u - 250uu_xu_{xx}u_{xxx}\beta_u - 225u_x^2u_{xx}^2\beta_u - 225u^2u^2u_{xxx}^2\beta_u - 30u_xu_{xx}u_{xxxxx}\beta_u - 30uu_{xxx}u_{xxxxx}\beta_u - u_{xxxxx}^2\beta_u + 45u^2\lambda_x + 45u^2u_x\lambda_u - 45u^2u_x\alpha_x - 45u^2u_x^2\alpha_u + 2025u^2u_x^2u_{xx}\beta_x + 675u^2u_xu_{xx}\beta_x + 675u^3u_{xxx}\beta_x + 45u^2u_{xxxxx}\beta_x + 2025u^4u_x^2\beta_u + 675u^2u_x^2u_{xx}\beta_u + 675u^3u_xu_{xxx}\beta_u - 45u^2u_xu_{xxxxx}\beta_u + 15u_{xx}\lambda_x + 15u_xu_{xx}\lambda_u - 15u_xu_{xx}\alpha_x - 15u_x^2u_{xx}\alpha_u + 675u^2u_xu_{xx}\beta_x + 225u_xu_{xx}^2\beta_x + 225uu_{xx}u_{xxx}\beta_x + 15u_{xx}uu_{xxxxx}\beta_x + 675u^2u_x^2u_{xx}\beta_u + 225u_x^2u_{xx}^2\beta_u + 225uu_xu_{xx}u_{xxx}\beta_u + 15u_xu_{xx}u_{xxxxx}\beta_u + 15u_x\lambda_{xx} + 30u_x^2\lambda_{ux} + 15u_xu_{xx}\lambda_u + 15u_x^3\lambda_{uu} - 15u_x^2\alpha_{xx} - 30u_x^3\alpha_{ux} - 15u_x^2u_{xx}\alpha_u - 15u_x^4\alpha_{uu} + 675u^2u_x^2\beta_{xx} + 225u_x^2u_{xx}\beta_{xx} + 225uu_xu_{xxx}\beta_{xx} + 15u_xu_{xxxxx}\beta_{xx} + 1350u^2u_x^3\beta_{ux} + 450u_x^3u_{xx}\beta_{ux} + 450uu_x^2u_{xxx}\beta_{ux} + 30u_x^2u_{xxxxx}\beta_{ux} + 675u^2u_x^2u_{xx}\beta_u + 225u_x^2u_{xx}^2\beta_u + 225uu_xu_{xx}u_{xxx}\beta_u + 15u_xu_{xx}u_{xxxxx}\beta_u + 675u^2u_x^4\beta_{uu} + 225u_x^4u_{xx}\beta_{uu} + 225uu_x^3u_{xxx}\beta_{uu} + 15u_x^3u_{xxxxx}\beta_{uu} - 30u_xu_{xx}\alpha_x - 30u_x^2u_{xx}\alpha_u - 30u_xu_{xt}\beta_x - 30u_x^2u_{xt}\beta_u + 15u\lambda_{xxx} + 45uu_x\lambda_{uxx} + 45uu_{xx}\lambda_{ux} + 15uu_{xxx}\lambda_u + 45uu_x^2\lambda_{uux} + 45uu_xu_{xx}\lambda_{uu} + 15uu_x^3\lambda_{uuu} - 5uu_x\alpha_{xxx} - 45uu_x^2\alpha_{uxx} - 45uu_xu_{xx}\alpha_{ux} - 15uu_xu_{xxx}\alpha_u - 45uu_x^3\alpha_{uux} - 45uu_x^2u_{xx}\alpha_{uu} - 15uu_x^4\alpha_{uuu} + 675u^3u_x\beta_{xxx} + 225uu_xu_{xx}\beta_{xxx} + 225u^2u_{xxx}\beta_{xxx} + 15uu_{xxxxx}\beta_{xxx} + 2025u^3u_x^2\beta_{uxx} + 675uu_x^2u_{xx}\beta_{uxx} + 675u^2u_xu_{xxx}\beta_{uxx} + 45uu_xu_{xxxxx}\beta_{uxx} + 2025u^3u_xu_{xx}\beta_{ux} + 675uu_xu_{xx}^2\beta_{ux} + 675u^2u_{xx}u_{xxx}\beta_{ux} + 45uu_{xx}u_{xxxxx}\beta_{ux} + 675u^3u_xu_{xxx}\beta_u + 225uu_xu_{xx}u_{xxx}\beta_u + 225u^2u_{xxx}^2\beta_u + 15uu_{xxx}u_{xxxxx}\beta_u + 2025u^3u_x^3\beta_{uux} + 675uu_x^3u_{xx}\beta_{uux} + 675u^2u_x^2u_{xxx}\beta_{uux} + 45uu_x^2u_{xxxxx}\beta_{uux} + 2025u^3u_x^2u_{xx}\beta_{uu} + 675uu_x^2u_{xx}^2\beta_{uu} + 675u^2u_xu_{xx}u_{xxx}\beta_{uu} + 45uu_xu_{xx}u_{xxxxx}\beta_{uu} + 675u^3u_x^4\beta_{uuu} + 225uu_x^4u_{xx}\beta_{uuu} + 225u^2u_x^3u_{xxx}\beta_{uuu} + 15uu_x^3u_{xxxxx}\beta_{uuu} - 45uu_{xx}\alpha_{uu} - 90uu_xu_{xx}\alpha_{ux} - 45uu_{xx}^2\alpha_u -$

$45uu_x^2u_{xx}\alpha_{uu}-45uu_{xt}\beta_{xx}-90uu_xu_{xt}\beta_{ux}-45uu_{xx}u_{xt}\beta_u-45uu_x^2u_{xt}\beta_{uu}-45uu_{xxx}\alpha_x-$

$45uu_xu_{xxx}\alpha_u-45uu_{xxt}\beta_x-45uu_xu_{xxt}\beta_u+\lambda_{xxxxx}+u_x(5\lambda_{uxxxx}-\alpha_{xxxxx})+u_{xx}(10\lambda_{uxxx}-$

$5\alpha_{xxxx})+u_{xxx}(10\lambda_{uxx}-5\alpha xxx)+u_{xxxx}(5\lambda_{ux}-5\alpha_{xx})+u_{xxxxx}(\lambda_u-5\alpha_x)+u_x^2(10\lambda_{uuxxx}-$

$5\alpha_{uxxxx}) + u_x^3(10\lambda_{uuuxx} - 10\alpha_{uuxxx}) + u_x^4(5\lambda_{[}uuuux - 10\alpha_{uuuxx}) + u_x^5(\lambda_{uuuuu} -$

$5\alpha_{uuuux})-u_x^6\alpha_{uuuuu}+u_{xx}^2(15\lambda_{uux}-30\alpha_{uxx})-15u_{xx}^3\alpha_{uu}+u_xu_{xx}(30\lambda_{uuxx}-30\alpha_{uxxx})+$

$u_xu_{xxx}(20\lambda_{uux}-25\alpha_{uxx})+u_xu_{xxxx}(5\lambda_{uu}-15\alpha_{ux})-6u_xu_{xxxxx}\alpha_u+u_{xx}u_{xxx}(10\lambda_{uu}-$

$35\alpha_{ux}) - 10u_{xx}u_{xxxx}\alpha_u + u_x^2u_{xx}(30\lambda_{uuux} - 60\alpha_{uuxx}) + u_xu_{xx}^2(15\lambda_{uuu} - 75\alpha_{uux}) +$

$u_x^2u_{xxx}(10\lambda_{uuu}-35\alpha_{uux})+u_x^3u_{xx}(10\lambda_{uuuu}-50\alpha_{uuux})-45u_xu_{xx}u_{xxx}\alpha_{uu}-10u_x^2u_{xxxx}\alpha_{uu}-$

$45u_x^2u_{xx}^2\alpha_{uuu}-15u_x^3u_{xxx}\alpha_{uuu}-15u_x^4u_{xx}\alpha_{uuuu}-5u_{xxx}^2\alpha_u+45u^2u_x\beta_{xxxxx}+15u_xu_{xx}\beta_{xxxxx}+$

$15uu_{xxx}\beta_{xxxxx} + u_{xxxxx}\beta_{xxxxx} + 225u^2u_x^2\beta_{uxxxx} + 75u_x^2u_{xx} + 75uu_xu_{xxx}\beta_{uxxxx}+$

$5u_xu_{xxxxx}\beta_{uxxxx} + 450u^2u_xu_{xx}\beta_{uxxx} + 150u_xu_{xx}^2\beta_{uxxx} + 150uu_{xx}u_{xxx}\beta_{uxxx}+$

$10u_{xx}u_{xxxxx}\beta_{uxxx} + 450u^2u_xu_{xxx}\beta_{uxx} + 150u_xu_{xx}u_{xxx}\beta_{uxx} + 150uu_{xxx}^2\beta_{uxx}+$

$10u_{xxx}u_{xxxxx}\beta_{uxx} + 225u^2u_xu_{xxxx}\beta_{ux} + 75u_xu_{xx}u_{xxxx}\beta_{ux} + 75uu_{xxx}u_{xxxxx}\beta_{ux}+$

$5u_{xxxx}u_{xxxxx}\beta_{ux}+45u^2u_xu_{xxxxx}\beta_u+15u_xu_{xx}u_{xxxxx}\beta_u+15uu_{xxx}u_{xxxxx}\beta_u+u_{xxxxx}^2\beta_u+$

$450u^2u_x^3\beta_{uuxxx} + 150u_x^3u_{xx}\beta_{uuxxx} + 150uu_x^2u_{xxx}\beta_{uuxxx} + 10u_x^2u_{xxxxx}\beta_{uuxxx}+$

$1350u^2u_x^2u_{xx}\beta_{uuxx} + 450u_x^2u_{xx}^2\beta_{uuxx} + 450uu_xu_{xx}u_{xxx}\beta_{uuxx} + 30u_xu_{xx}u_{xxxxx}\beta_{uuxx}+$

$675u^2u_xu_{xx}^2\beta_{uux} + 225u_xu_{xx}^3\beta_{uux} + 225uu_{xx}^2u_{xxx}\beta_{uux} + 15u_{xx}^2u_{xxxxx}\beta_{uux}+$

$900u^2u_x^2u_{xxx}\beta_{uux} + 300u_x^2u_{xx}u_{xxx}\beta_{uux} + 300uu_xu_{xxx}^2\beta_{uux} + 20u_xu_{xxx}u_{xxxxx}\beta_{uux}+$

$450u^2u_xu_{xx}u_{xxx}\beta_{uu} + 150u_xu_{xx}^2u_{xxx}\beta_{uu} + 150uu_{xx}u_{xxx}^2\beta_{uu} + 10u_{xx}u_{xxx}u_{xxxxx}\beta_{uu}+$

$225u^2u_x^2u_{xxxx}\beta_{uu} + 75u_x^2u_{xx}u_{xxxx}\beta_{uu} + 75uu_xu_{xxx}u_{xxxx}\beta_{uu} + 450u^2uu_x^4\beta_{uuuxx}+$

$150u_x^4u_{xx}\beta_{uuuxx} + 150uu_x^3u_{xxx}\beta_{uuuxx} + 10u_x^3u_{xxxxx}\beta_{uuuxx} + 1350u^2u_x^3u_{xx}\beta_{uuux}+$

$450u_x^3u_{xx}^2\beta_{uuux} + 450uu_x^2u_{xx}u_{xxx}\beta_{uuux} + 30u_x^2u_{xx}u_{xxxxx}\beta_{uuux} + 675u^2u_x^2u_{xx}^2\beta_{uuu}+$

$225u_x^2u_{xx}^3\beta_{uuu} + 225uu_xu_{xx}^2u_{xxx}\beta_{uuu} + 15u_xu_{xx}^2u_{xxxxx}\beta_{uuu} + 450u^2u_x^3u_{xxx}\beta_{uuu}+$

$150u_x^3u_{xx}u_{xxx}\beta_{uuu} + 150uu_x^2u_{xxx}^2\beta_{uuu} + 10u_x^2u_{xxx}u_{xxxxx}\beta_{uuu} + 225u^2u_x^5\beta_{uuuux}+$

$75u_x^5u_{xx}\beta_{uuuux} + 75uu_x^4u_{xxx}\beta_{uuuux} + 5u_x^4u_{xxxxx}\beta_{uuuux} + 450u^2u_x^4u_{xx}\beta_{uuuu}+$

$150u_x^4u_{xx}^2\beta_{uuuu} + 150uu_x^3u_{xx}u_{xxx}\beta_{uuuu} + 10u_x^3u_{xx}u_{xxxxx}\beta_{uuuu} + 45u^2u_x^6\beta_{uuuuu}+$

$15u_x^6u_{xx}\beta_{uuuuu} + 15uu_x^5u_{xxx}\beta_{uuuuu} + u_x^5u_{xxxxx}\beta_{uuuuu} - 5u_{xt}\beta_{xxxx} - 20u_xu_{xt}\beta_{uxxx} -$

$30u_{xx}u_{xt}\beta_{uxx} - 20u_{xxx}u_{xt}\beta_{ux} - 5u_{xxxx}u_{xt}\beta_u - 30u_x^2u_{xt}\beta_{uuxx} - 60u_xu_{xx}u_{xt}\beta_{uux} -$

$15u_{xx}^2u_{xt}\beta_{uu} - 20u_xu_{xxx}u_{xt}\beta_{uu} - 20u_x^3u_{xt}\beta_{uuux} - 30u_x^2u_{xx}u_{xt}\beta_{uuu} - 5u_x^4u_{xt}\beta_{uuuu} -$

$5u_{xxt}\beta_{xxx} - 5u_x u_{xxt}\beta_{uxx} - 15u_{xx}u_{xxt}\beta_{ux} - 5u_{xxx}u_{xxt}\beta_u - 15u_x^2 u_{xxt}\beta_{uux} - 15u_x u_{xx}u_{xxt}\beta_{uu} -$
$5u_x^3 u_{xxt}\beta_{uuu} - 5u_{xxxt}\beta_{xx} - 10u_x u_{xxxt}\beta_{ux} - 5u_{xx}u_{xxxt}\beta_u - 5u_x^2 u_{xxxt}\beta_{uu} - 5u_{xxxxt}\beta_x -$
$$5u_x u_{xxxxt}\beta_u = 0 \qquad (4.19)$$

Since α, β and λ are functions of x, t and u only, we equated the coefficients of the powers of $u = u(x,t)$ and their combinations to zero.

We obtained the determining equations as follows.

Table 4.1: Determining Equations for Sawada-Kotera Equations

Monomials	**Equations**	**Equation number**
$u_x u_{xxxxt}$	$-5\beta_u = 0$	(i)
u_{xxxxt}	$-5\beta_x = 0$	(ii)
u_{xxx}^2	$-\beta_u = -\beta_u$	(iii)
$u_{xx}u_{xxx}u_{xxxxx}$	$\beta_{uu} = 0$	(iv)
$u_x u_{xxxxx}$	$-6\alpha_u = -5\beta_{uxxxx} - \alpha_u - 15\beta_{xx}$	(v)
u_{xxxxx}	$\beta_{xxxxx} - \lambda_u + \beta_t +_u -5\alpha_x = 0$	(vi)
$u_{xx}u_{xxx}$	$10\lambda_{uu} - 35\alpha_{ux} = 0$	(vii)
u_{xx}^3	$\alpha_{uu} = 0$	(viii)
u_x	$-\alpha_t = \alpha_{xxxxx} - 5\lambda_{uxxxx} - 15\lambda_{xx}$	(ix)

The subscripts indicate the derivatives. Thus the solution of the determining equations is elementary.

Equations (i) and (ii) shows that β is a function of t only since it is independent of u and x . So $\beta = \beta(t)$. Equation (v) shows that α does not depend on u since $\beta_{uxxxx} = 0$ and $\beta_{xx} = 0$ and $\alpha = \alpha(x,t)$ thus $\alpha_{xx} = 0$ which shows that α is linear in x. Therefore $\alpha = c(t)x + d(t)$

Equation (vi) also shows that $\alpha_x = \frac{1}{5}\beta_t$ implying that $\alpha = \frac{1}{5}\beta_t x + \mu(t)$ where μ is some functions of t only.

Equating the values of α we have

$\alpha = c(t)x + d(t)$ and $\alpha = \frac{1}{5}\beta_t x + \mu(t)$ (4.20)

to obtain

$c(t) = \frac{1}{5}\beta_t$ and $d(t) = \mu(t)$ thus we have the values of α as given below

$\alpha = a_1 + \frac{1}{5}a_3 x$ (a)

By equation (vii), we see that $\lambda_{uu} = 0$ because α is not a function of u . Thus λ is linear in u .

So $\lambda = \tau(x,t)u + \psi(x,t)$ (4.21)

for certain functions of τ and ψ.

Referring to equation (ix) we have $-\alpha_t = \alpha_{xxxxx} - 5\lambda_{uxxxx} - 15\lambda_{xx}$ then $\lambda_{xx} = 0$ and $\alpha_{xxxxx} = 0$ since $\alpha_{xx} = 0$ thus $-\alpha_t = -5\lambda_{uxxxx}$

$\alpha_t = 5\lambda_{uxxxx} = 5\tau_x$ (4.22)

But $\alpha = c(t)x + d(t)$ so

$\tau_x = \frac{1}{5}[\alpha_t] = \frac{1}{5}[c_t(t)x + d_t(t)] = \frac{1}{5}[\frac{1}{5}\beta_{tt}x + \mu_t(t)]$

Thus we have

$\tau_x = \frac{1}{5}[\alpha_t] = \frac{1}{5}[\frac{1}{5}\beta_{tt}x + \mu_t(t)]$ (4.23)

And also $\tau_x = \frac{1}{5}[\alpha_t] = \frac{1}{5}[c_t(t)x+d_t(t)]$ (4.24)

Integrating both (4.23) and (4.24) respectively, we had

$$\tau = \frac{1}{5}[\frac{1}{10}\beta_{tt}x^2 + \mu_t(t)x] + \eta(t)$$

$$= \frac{1}{50}\beta_{tt}x^2 + \frac{1}{5}\mu_t(t)x + \eta(t) \quad (4.25)$$

$$\tau = \frac{1}{10}c_t(t)x^2 + \frac{1}{5}d_t(t)x + \eta(t) \quad (4.26)$$

Lastly equation (x) implied that τ and ψ be the solutions of Sawada-Kotera equation.

$$\lambda = \tau(x,t)u + \psi(x,t) \quad (4.27)$$

So we have

$$\lambda_t = \tau_t(x,t)u + \psi_t(x,t) \quad (4.28)$$

$$\lambda_{xxxxx} = \tau_{xxxxx}(x,t)u + \psi_{xxxxx}(x,t) \quad (4.29)$$

Therefore from equation (x) we had

$$\tau_t(x,t)u + \psi_t(x,t) = -(\tau_{xxxxx}(x,t)u + \psi_{xxxxx}(x,t)) \quad (4.30)$$

Equating the coefficients of u and other terms, we obtained

$$\tau_t(x,t) = -\tau_{xxxxx}(x,t) \tag{4.31}$$

and

$$\psi_t(x,t) = -\psi_{xxxxx}(x,t) \tag{4.32}$$

Using equations (4.25) and (4.26) upon equations (4.31) and (4.32), we obtained

$$\tau_t = \tfrac{1}{50}\beta_{ttt}x^2 + \tfrac{1}{5}\mu_{tt}(t)x + \eta_t(t) \tag{4.33}$$

$$\tau_t = \tfrac{1}{10}c_{tt}(t)x^2 + \tfrac{1}{5}d_{tt}(t)x + \eta_t(t) \tag{4.34}$$

So upon differentiating equations (4.33) and (4.34) with respect to x , we had

$-\tau_{xxxxx} = \frac{1}{25}\beta_{tt}$ or $-\tau_{xxxxx} = \frac{1}{5}c_t$

Therefore, we wrote the equations as

$\frac{1}{50}\beta_{ttt}x^2 + \frac{1}{5}\mu_{tt}(t)x + \eta_t(t) = -\frac{1}{25}\beta_{tt}$

$\frac{1}{10}c_{tt}(t)x^2 + \frac{1}{5}d_{tt}(t)x + \eta_t(t) = -\frac{1}{5}c_t$

Equating the coefficients of x , we got $c_{tt} = 0, d_{tt} = 0, \beta_{ttt} = 0, \mu_{tt} = 0$

Thus we had $\eta_t(t) = -\frac{1}{25}\beta_{tt}$ and also $\eta_t(t) = -\frac{1}{5}c_t$

Hence β is linear in t thus it can be expressed as

$$c(t) = c_0 + c_1 t \tag{4.35}$$

and $d(t) = d_0 + d_1 t$ (4.36)

therefore we express β as

$$\beta = a_2 + ta_3 \tag{b}$$

Finally, with $\lambda = \tau(x,t)u + \psi(x,t)$ then we have

$$\lambda = -\tfrac{2}{5}ua_3 \tag{c}$$

Thus with $G = \alpha(x,t,u)\frac{\partial}{\partial x} + \beta(x,t,u)\frac{\partial}{\partial t} + \lambda(x,t,u)\frac{\partial}{\partial u}$ we had the general solutions from the determining equations became

$$\alpha = a_1 + \tfrac{1}{5}a_3 x \tag{4.37i}$$

$$\beta = a_2 + ta_3 \tag{4.37ii}$$

$$\lambda = -\tfrac{2}{5}ua_3 + \xi(x,t) \tag{4.37iii}$$

where ξ is an arbitrary solution of Sawada-Kotera equation.

4.4 Infinitesimal Generators and Lie Groups

The infinitesimal transformations of Sawada-Kotera equation, α, β and λ are expressed as

	w_1	w_2	w_3
	$\downarrow$	$\downarrow$	$\downarrow$
$\alpha =$	$1.c_1$	$0.c_2$	$\frac{1}{5}.c_3x$
$\beta =$	$0.c_1$	$1.c_2$	$1.c_3t$
$\lambda =$	$0.c_1$	$0.c_2$	$-\frac{2}{5}.c_3u$

Thus we formed the corresponding basis/ infinitesimal generators as follows.

$w_1 = \frac{\partial}{\partial x}$

$w_2 = \frac{\partial}{\partial t}$

$w_3 = x\frac{\partial}{\partial x} + 5t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u}$

We then computed the Lie brackets of the vector fields of the infinitesimal symmetry (w_i) by using

$[w_i, w_j] = w_iw_j - w_jw_i$

to obtain

w_i, w_j	w_1	w_2	w_3
w_1	0	0	w_1
w_2	0	0	$5w_2$
w_3	$-w_1$	$-5w_2$	0

Lie groups admitted by infinitesimal generators were obtained by solving the corresponding Lie equations through exponentiation which led to the formation of the groups as follows

$w_1 = \frac{\partial}{\partial x}; G_1(\varepsilon) : X(x, t, u; \varepsilon) \to X_1(x + \varepsilon, t, u)$

$w_2 = \frac{\partial}{\partial t}; G_2(\varepsilon) : X(x,t,u;\varepsilon) \to X_2(x, t+\varepsilon, u)$

$w_3 = x\frac{\partial}{\partial x} + 5t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u}; G_3(\varepsilon) : X(x,t,u;\varepsilon) \to X_3(xe^{\varepsilon}, te^{5\varepsilon}, ue^{-2\varepsilon})$

Where G_1 and G_2 are trivial groups since they are translation and scaling while G_3 is a non-trivial group.

4.5 Group Transformations of Solutions

If each G_i is a symmetry group and $u = \rho(x,t)$ is a solution of Sawada-Kotera equation (4.1), then the functions $\overline{u_j}$ below are also solutions.[28]

$\overline{u_1} = \rho(x-\varepsilon, t)$

$\overline{u_2} = \rho(x, t-\varepsilon)$

$\overline{u_3} = \rho(xe^{-\varepsilon}, te^{-5\varepsilon})e^{-2\varepsilon}$

Noting that groups G_1 and G_2 are trivial groups since they are translation and scaling while G_3 is a non-trivial group.

4.6 Invariant Solutions and Exact Power Series Solutions

A group invariant solution is obtained when a group of transformations maps a solution into itself. The invariant solution of equation (4.1) under the one -parameter group of generator V can be obtained by calculating two independent invariants $N_1 = k(x,t)$ and $N_2 = \mu(x,t,u)$ by solving the equation

$$N(J) \equiv \alpha(x,t,u)\frac{\partial N}{\partial x} + \xi(x,t,u)\frac{\partial N}{t} + \rho(x,t,u)\frac{\partial N}{\partial u} = 0 \quad (4.39)$$

Or its system of characteristics

$$\frac{dx}{\alpha(x,t,u)} = \frac{dt}{\xi(x,t,u)} = \frac{du}{\rho(x,t,u)} \quad (4.40)$$

Here we consider the group transformations that arise from all the generators of (4.1)

We then allocate one of the invariants as a function of the other as given below

$$\mu = \phi(k) \quad (4.41)$$

We then substitute for μ , in (4.41) to get an ordinary differential equation for the function $\phi(k)$ of one variable. By doing this we decrease the figure/number of independent variables by one.

We now show the list of generators (X_i) and their equivalent Invariant Solutions (u)

Case 1

For the infinitesimal generator $w_2 = \frac{\partial}{\partial x}$, the invariant solution under transformation has a system of characteristics $\frac{dt}{0} = \frac{dx}{1}$

Integrating the equation, we obtained

$t = \mu, t = \xi$ and $u = \phi(t)$.

When we substituted $u_t = \rho', u_x = 0$ for $\phi' = \frac{d\phi}{dt}$ into equation (4.1) we got the trivial solution to be $u = \phi(t) = c$ (4.42)

Case 2

For the generator $w_2 = \frac{\partial}{\partial t}$, the invariant solution under transformation has a system of characteristics $\frac{dt}{1} = \frac{dx}{0}$

Integrating the equation, we have $x = \mu, x = \xi$ and $u = \rho(x)$

When we substituted $u_t = 0, u_x = \phi', u_{xx} = \phi'', u_{xxx} = \phi''', u_{xxxxx} = \phi^{(5)}$ for $\phi' = \frac{d\phi}{dx}$ into equation (4.1), the equation was reduced into the following ordinary differential equation

$$45\phi^2\phi' + 15\phi'\phi'' + 15\phi\phi''' + \phi^{(5)} = 0 \tag{4.43}$$

where $\phi' = \frac{d\phi}{d\mu}$.

Case 3

For the generator, $w_3 = x\frac{\partial}{\partial x} + 5t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u}$, the invariant solution under transformation has a system of characteristics $\frac{dt}{5t} = \frac{dx}{x} = \frac{du}{-2u}$

Integrating the equation yielded

$lnx = lnt^{\frac{1}{5}} + c \Rightarrow \frac{x}{t^{\frac{1}{5}}} \Rightarrow \mu = xt^{-\frac{1}{5}}$

and

$lnx = lnu^{-\frac{1}{2}} + c \Rightarrow \frac{x}{u^{-\frac{1}{2}}} \Rightarrow \mu = xu^{\frac{1}{2}}$

Giving $xt^{-\frac{1}{5}} = xu^{\frac{1}{2}} \Rightarrow t^{-\frac{1}{5}} = u^{\frac{1}{2}}$ and on squaring both sides, we got $u = t^{-\frac{2}{5}}\phi(\mu)$ where $\mu = xt^{-\frac{1}{5}}$.

Substituting into equation (4.1), reduced the equation into the following Ordinary Differential Equation

$$-\frac{2}{5}\phi - \frac{1}{5}\mu\phi' + 45\phi^2\phi' + 15\phi'\phi'' + 15\phi\phi''' + \phi^{(5)} = 0 \tag{4.44}$$

where $\phi' = \frac{d\phi}{d\mu}$

In this case, the exact solutions to the Sawada-Kotera equation were obtained from some ODEs or from PDEs of lower order than the original PDE [21].

Besides this, we want to identify the explicit solutions conveyed in terms of elementary or, at least, known functions of mathematical physics, in terms of quadratures. This is not always the case, for simple semilinear PDEs. However, we know that the power series can be used to solve differential equations, including many complicated differential equations with non-constant coefficients.

We considered the exact analytic solutions to the reduced equations using the power series method. Once we obtained the exact analytic solutions of the reduced equations (ODEs), the exact power series solutions to the original PDEs were obtained.

In this case, we considered equations (4.43) and (4.44).

In view of (4.43) we have

$45\phi^2\phi' + 15\phi'\phi'' + 15\phi\phi''' + \phi^{(5)} = 0$

We obtain its solution by use of a power series method given as

$$\phi(\beta) = \Sigma_{a=0}^{\infty} c_a\beta^a \tag{4.45}$$

Substituting (4.45) into (4.43) we got

$120c_5 + \Sigma_{a=1}^{\infty}(a+1)(a+2)(a+3)(a+4)(a+5)c_{a+5}\beta^a + 90c_0c_3 + 15\Sigma_{z=0}^{a}(a-z+1)(a-z+2)(a-z+3)c_zc_{a-z+2}\beta^a + 30c_1c_2 + 15\Sigma_{z=0}^{a}(a-z+1)(a-z+2)(z+1)c_{z+1}c_{a-z+2} +$

$45c_0^2c_1 + 45\Sigma_{z=0}^{a}\Sigma_{i=0}^{z}(a-z+1)c_ic_{z-i}c_{a-z+1}$ (4.46)

Setting $a = 0$, we got

$120c_5 + 90c_0c_3 + 30c_1c_2 + 45c_0^2c_1 = 0$

On simplifying we get

$c_5 = -\frac{1}{120}(90c_0c_3 + 30c_1c_2 + 45c_0^2c_1)$ (4.47)

For $a \geq 1$ we obtained

$c_{a+5} = \frac{-1}{(a+1)(a+2)(a+3)(a+4)(a+5)}[15\Sigma_{z=0}^{a}(a-z+1)(a-z+2)(a-z+3)c_zc_{a-z+3} +$

$15\Sigma_{z=0}^{a}(a-z+1)(a-z+2)(z+1)c_{z+1}c_{a-z+2} + 45\Sigma_{z=0}^{a}\Sigma_{i=0}^{z}(a-z+1)c_ic_{z-i}c_{a-z+1}]$ (4.48)

for $a - 0, 1, 2...$

When $a = 1$ we had

$c_6 = \frac{-1}{720}(360c_0c_4 + 180c_1c_3 + 90c_0^2c_2 + 60c_2^2 + 90c_0c_1^2)$ (4.49)

When $a = 2$ we had

$c_7 = \frac{-1}{2520}(900c_0c_5 + 540c_1c_4 + 135c_0^2c_3 + 360c_2c_3 + 270c_0c_1c_2 + 45c_1^3$ (4.50)

Thus the power series solution (4.45) into (4.43) gave an exact analytic solution of the form:

$\phi(\beta) = c_0 + c_1\beta + c_2\beta^2 + c_3\beta^3 + +c_4\beta^4 + c_5\beta^5 + \Sigma_{a=1}^{\infty}c_{a+5}\beta^{a+5}$

$= c_0 + c_1\beta + c_2\beta^2 + c_3\beta^3 + c_4\beta^4 - \frac{1}{120}(90c_0c_3 + 30c_1c_2 + 45c_0^2c_1)\beta^5 -$

$\Sigma_{a=1}^{\infty}\frac{1}{(a+1)(a+2)(a+3)(a+4)(a+5)}[15\Sigma_{z=0}^{a}(a-z+1)(a-z+2)(a-z+3)c_zc_{a-z+3} +$

$15\Sigma_{z=0}^{a}(a-z+1)(a-z+2)(z+1)c_{z+1}c_{a-z+2} + 45\Sigma_{z=0}^{a}\Sigma_{i=0}^{z}(a-z+1)c_ic_{z-i}c_{a-z+1}]\beta^{a+5}$ (4.51)

Now, the exact power series solution of (4.1) was obtained to be

$u(x,t) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + \Sigma_{a=1}^{\infty}c_{a+5}x^{a+5}$

$= c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 - \frac{1}{120}(90c_0c_3 + 30c_1c_2 + 45c_0^2c_1)x^5 -$

$\Sigma_{a=1}^{\infty}\frac{1}{(a+1)(a+2)(a+3)(a+4)(a+5)}[15\Sigma_{z=0}^{a}(a-z+1)(a-z+2)(a-z+3)c_zc_{a-z+3} +$

$15\Sigma_{z=0}^{a}(a-z+1)(a-z+2)(z+1)c_{z+1}c_{a-z+2} + 45\Sigma_{z=0}^{a}\Sigma_{i=0}^{z}(a-z+1)c_ic_{z-i}c_{a-z+1}]x^{a+5}$ (4.52)

where $c_i (i = 0, 1, 2, 3, 4)$ are arbitrary constants.

Also, we found a solution of equation (4.44) in a power series method of the form (4.45). substituting into (4.44) and comparing the coefficients, we obtained

$120c_5 + \Sigma_{a=1}^{\infty}(a+1)(a+2)(a+3)(a+4)(a+5)c_{a+5}\beta^a + 90c_0c_3 + 15\Sigma_{z=0}^{a}(a - z + 1)(a-z+2)(a-z+3)c_zc_{a-z+3}\beta^a + 30c_1c_2 + 15\Sigma_{z=0}^{a}(a-z+1)(a-z+$

$2)(z+1)c_z+1c_{a-z+2}+45c_0^2c_1+45\Sigma_{z=0}^{a}\Sigma_{i=0}^{z}(a-z+1)c_ic_{z-i}c_{a-z+1}-\frac{2}{5}c_a-\frac{1}{5}ac_a=$

0 (4.53)

Setting the coefficients for $a=0$ we obtained

$120c_5+90c_0c_3+30c_1c_2+45c_0^2c_1-\frac{2}{5}c_0=0$

which was simplified to give

$$c_5=-\frac{1}{120}(90c_0c_3+30c_1c_2+45c_0^2c_1-\frac{2}{5}c_0) \quad (4.54)$$

For $a\geq 1$, we obtained

$c_{a=5}=\frac{-1}{(a+1)(a+2)(a+3)(a+4)(a+5)}[15\Sigma_{z=0}^{a}(a-z+1)(a-z+2)(a-z+3)c_zc_{a-z+3}+$

$15\Sigma_{z=0}^{a}(a-z+1)(a-z+2)(z+1)c_{z+1}c_{a-z+2}+45\Sigma_{z=0}^{a}\Sigma_{i=0}^{z}(a-z+1)c_ic_{z-i}c_{a-z+1}-$

$\frac{2}{5}c_a-\frac{1}{5}ac_a]$ (4.55)

For the values of $a=0,1,2...$

When $a=1$ we have

$$c_6=\frac{1}{720}(360c_0c_4+180c_1c_3+90c_0^2c_2+60c_2^2+90c_0c_1^2-\frac{3}{5}c_1 \quad (4.56)$$

When $a=2$ we have

$$c_7=\frac{-1}{2520}(900c_0c_5+540c_1c_4+135c_0^2c_3+360c_2c_3+270c_0c_1c_2+45c_1^3-\frac{4}{5}c_2 \quad (4.57)$$

Therefore the power series solution of equation (4.44) is given as

$\phi(\beta)=c_0+c_1\beta+c_2\beta^2+c_3\beta^3+c_4\beta^4+c_5\beta^5+\Sigma_{a=1}^{\infty}c_{a+5}\beta^{a+5}$

$=c_0+c_1\beta+c_2\beta^2+c_3\beta^3+c_4\beta^4-\frac{1}{120}(90c_0c_3+30c_1c_2+45c_0^2c_1-\frac{2}{5}c_0)\beta^5-$

$\Sigma_{a=1}^{\infty}\frac{1}{(a+1)(a+2)(a+3)(a+4)(a+5)}[15\Sigma_{z=0}^{a}(a-z+1)(a-z+2)(a-z+3)c_zc_{a-z+3}+$

$15\Sigma_{z=0}^{a}(a-z+1)(a-z+2)(z+1)c_{z+1}c_{a-z+2}+45\Sigma_{z=0}^{a}\Sigma_{i=0}^{z}(a-z+1)c_ic_{z-i}c_{a-z+1}-$

$\frac{2}{5}c_a-\frac{1}{5}ac_a]\beta^{a+5)}$

Hence the exact analytic solution to equation (4.1) is given as

$u(x,t)=c_0t^{-\frac{2}{5}}+c_1xt^{-\frac{3}{5}}+c_2x^2t^{-\frac{4}{5}}+c_3x^3t^{-1}+c_4x^4t^{-\frac{6}{5}}-\frac{1}{120}(90c_0c_3+30c_1c_2+$

$45c_0^2c_1-\frac{2}{5}c_0)x^5t^{-\frac{7}{5}}-\Sigma_{a=1}^{\infty}\frac{1}{(a+1)(a+2)(a+3)(a+4)(a+5)}[15\Sigma_{z=0}^{a}(a-z+1)(a-z+2)(a-$

$z+3)c_zc_{a-z+3}+15\Sigma_{z=0}^{a}(a-z+1)(a-z+2)(z+1)c_{z+1}c_{a-z+2}+45\Sigma_{z=0}^{a}\Sigma_{i=0}^{z}(a-z+$

$1)c_ic_{z-i}c_{a-z+1}-\frac{2}{5}c_a-\frac{1}{5}ac_a]x^{a+5}t^{-\frac{a+7}{5}}$ (4.58)

4.7 Symmetry Solutions

Symmetry transformations convert known solutions into new solutions. Considering group transformations that arise from the infinitesimal generators

$w_1 = \frac{\partial}{}, w_2 = \frac{\partial}{}, w_3 = x\frac{\partial}{} + 5t\frac{\partial}{} - 2u\frac{\partial}{}$

known to be

$G_1 : X(x, t, u; \varepsilon) \to X_1(x + \varepsilon, t, u)$

$G_2 : X(x, t, u; \varepsilon) \to X_2(x, t + \varepsilon, u)$

$G_3 : X(x, t, u; \varepsilon) \to X_3(xe^{\varepsilon}, te^{5\varepsilon}, ue^{-2\varepsilon})$

Since $u = u(x, t)$ is a known solution of equation (4.1) and so is

$G_3(\varepsilon)f(x, t) = f(xe^{-\varepsilon}, te^{-5\varepsilon)e^{-2\varepsilon}}$

We consider the group G_3. Thus the new symmetry transformed solution under G_3 becomes

$$u = f(xe^{\varepsilon}, te^{5\varepsilon}).e^{-2} \tag{4.59}$$

whenever a known solution of (4.1)is given as $u = u(x, t)$

Solution 1

Considering the invariant result of (4.1), $u = c$ and substituting into equation (4.59) we obtain

$u = ce^{-2\varepsilon}$

Solution 2

Inserting the exact solution

$u = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 - \frac{1}{120}(90c_0c_3 + 30c_1c_2 + 45c_0^2c_1)x^5 - (b^*)x^{a+5}$

as a known solution of equation (4.1) in which

$b^* = \Sigma_{a=1}^{\infty}\frac{1}{(a+1)(a+2)(a+3)(a+4)(a+5)}[15\Sigma_{z=0}^{a}(a - z + 1)(a - z + 2)(a - z + 3)c_zc_{a-z+3} + 15\Sigma_{z=0}^{a}(a - z + 1)(a - z + 2)(z + 1)c_{z+1}c_{a-z+2} + 45\Sigma_{z=0}^{a}\Sigma_{i=0}^{z}(a - z + 1)c_ic_{c-i}c_{a-z+1}]$

into (4.59) then we obtain

$u = [c_0+c_1xe^{\varepsilon}+c_2(xe^{\varepsilon})^2+c_3(xe^{\varepsilon})^3+c_4(xe^{\varepsilon})^4-\frac{1}{120}(90c_0c_3+30c_1c_2+45c_0^2c_1)(xe^{\varepsilon})^5-(b^*)(xe^{(a\varepsilon+5\varepsilon)})].e^{-2\varepsilon}$

Solution 3

Substituting the exact solution

$u = c_0t^{-\frac{2}{5}} + c_1xt^{-\frac{3}{5}} + c_2x^2t^{-\frac{4}{5}} + c_3x^3t^{-1} + c_4x^4t^{-\frac{6}{5}} - \frac{1}{120}(90c_0c_3 + 30c_1c_2 + 45c_0^2c_1 - \frac{2}{5}c_0)x^5t^{-\frac{7}{5}} - (b^*)(x^{a+5}t^{-\frac{a+7}{5}})$

as a known solution of equation (4.1) and b^* taken as stated above into equation (4.59) we obtain

$u = [c_0(te^{5\varepsilon})^{-\frac{2}{5}}+c_1xe^{\varepsilon}(te^{5\varepsilon})^{-\frac{3}{5}}+c_2(xe^{\varepsilon})^2(te^{5\varepsilon})^{-\frac{4}{5}}+c_3(xe^{\varepsilon})^3(te^{5\varepsilon})^{-1}+c_4(xe^{\varepsilon})^4(te^{5\varepsilon})^{-\frac{6}{5}}-\frac{1}{120}(90c_0c_3 + 30c_1c_2 + 45c_0^2 - \frac{2}{5}c_0)(xe^{\varepsilon})^5(te^{5\varepsilon})^{-\frac{7}{5}} - (b^*)(xe^{\varepsilon})^{a+5}(te^{5\varepsilon})^{-\frac{a+7}{5}})].e^{-2\varepsilon}$

Solutions 1 and 2 are trivial solutions since they are generated from trivial groups; G_1 and G_2. Solution 3 is a non-trivial solution from the non-trivial group; G_3. Thus the new symmetry solutions were successfully obtained.

CHAPTER FIVE

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

5.1 Introduction

This section entails summary, conclusions and recommendations.

5.2 Summary

In this study, we have obtained the symmetries and similarity reductions of the Sawada-Kotera equation which is highly nonlinear using Lie symmetry analysis method. We developed infinitesimal transformations, prolongations (extensions of the generator), symmetry generators and group transformations of the equation. All the group-invariant solutions to the equations are considered and the exact analytic solutions are investigated by using the power series method. We also obtained symmetry solutions of Sawada-Kotera equation from the exact power series solutions.

5.3 Conclusions

Our obtained symmetry solutions demonstrate that Lie symmetry analysis method is straightforward and best mathematical tool to obtain analytical solutions of highly nonlinear PDE's.

The thesis has proved that nonlinear differential equations can be solved easily to obtain their exact solutions which has direct impact on the big four agenda in terms of manufacturing as can be expressed in modeling of mechanical waves, water waves, sound waves, light waves and more so in navigation.

5.4 Recommendations

We hope that this method can be more effectively used to investigate others NLEEs which are frequently used in applied mathematics, physical sciences and engineering.

The solution can be obtained more easily if solvers like MATHEMATICA, MATLAB and MATHTYPE can be involved since the working is so rigorous and time consuming.

5.5 Suggestions for Further Research

Future research may take into consideration the solutions of the sixth and higher order nonlinear partial differential equations that have not been determined in previous researches.

REFERENCES

1. Abdul-Majid, W. (2011). Multiple soliton solution for (2+1)-dimensional Sawada-Kotera and Caudrey-Dodd-Gibbon equations. Mathematical Methods in the Applied Sciences, 34.

2. Aminer, T. (2014). Lie symmetry solution of fourth order nonlinear Ordinary Differential Equation. PhD thesis, Jaramogi Oginga Odinga University of Science and Technology,Kenya.

3. Aminus, M. (2014). Symmetry Analysis and Invariants Solutions of Laplace Equation on Surfaces of Revolution. *Advance Mathematical and Science Journal* 3, 23-31 .

4. Andronikos, P., Richard, M., and Peter, G. (2016). Lie Symmetries of (1+2) Nonautonomous Evolution Equations in Financial Mathematics.Mathematics,4, 34.

5. Bluman, G., and Anco, S. (2002). *Symmetry and Integration Methods for Differential Equations*. Springer New York, USA.

6. Bluman, G., and Kumei, S. (1989). *Symmetries and Differential Equations*. Springer- Verlag: New York, NY, USA.

7. Cantwell, B. (2002). Introduction to Symmetry Analysis. Cambridge: Cambridge University Press.

8. Dai Z., and Liu C. (2008). Exact soliton solutions for the fifth order Sawada-Kotera equation. *Applied Mathematics and Computation*, 206, 272-275.

9. Dingjiang, H., Xiangxiang, L., and Shunchang, Y. (2017). Lie Symmetry Classification of the Generalised Nonlinear Beam Equation. Symmetry, 9, 1-15.

10. Dresner, L. (1999). Application of Lie's Theory of Ordinary and Partial Differential Equations. London: Institute of Physics.

11. Fritz, S. (2000). Solving Second Order Differential Equations with Lie Symmetries. Kluwer Academic Publishers, 39-113.

12. Gang, W. W., Tian, Z. X., and Tao, F. (2014). Lie Symmetry Analysis and Explicit Solutions of the Time Fractional Fifth-order Korteweg-De Vries equations. 9, 1-6.

13. George, W., and Gregory, J. R. (1988). New Symmetries for Ordinary Differential Equations. Oxford University Press, 87-94.

14. Hasibun, N., Farah, A. and Syed, T. -D (2013). Extended generalised Riccati equation mapping methods for the fifth order Sawada- Kotera equation. *American Institute of Physics*, Advances, 3.

15. He, G.-Liu., and Geng, X.-G. (2012). An extension for the modified Sawada-Kotera equation and conservation laws. Chinese Physics B, IOP Publishing Ltd, 21.

16. Hui, M., Q.P., Liu., and Lingling, X. (2018). Supersymmetric Sawada-Kotera Equation:Backlund-Darboux Transformations and Applications. *Journal of Nonlinear Mathematical Physics*, 25, 375-386.

17. Inc, M., Kilic, B., Karatas, E., and Akgul, A. (2017). Solitary Wave Solutions for the Sawada-Kotera Equation. *Journal of Advanced Physics*, 6, American Scientific Publishers, 288-293.

18. Islam, S., Khan, K., and Akbar, M. (2015). Exact Solutions of Unsteady Kortweg-de Vries and Time Regularised Long Wave Equations. Springerplus.

19. Janet, M. (1920). Systems of Linear Partial Differential Equations. *Journal of Mathematics*,83, 65-123.As cited by Fritz, S. (2000). in Solving Second Order Differential Equations with Lie Symmetries.

20. Juan, B., Victor, P., Vadym, V., and Pedro, J. (2018). Lie Symmetries, Qualitative Analysis and Exact Solutions of Nonlinear Schrodinger Equations with inhomogeneous nonlinearities. Discrete and continuous Dynamical Systems,9, 221-233.

21. Khongorzul, D., Hiroyuki, O., and Uuganbayar, Z. (2018). Lie Symmetry Analysis of a class of time fractional nonlinear evolution systems. 1-15.

22. Loewy, A. (1906). Decomposing Linear Differential Equations into components of lowest order. *Mathematics Annual Journal.*,56, Elsevier, 89-117.

23. Manoj, G., and Karanjeet, S. (2017). Symmetry Analysis of Time-Fractional Potential Burgers' Equation. *Mathematical Communications*,22, 1-11.

24. Nucci, M., Cerquetelli, T., and Ciccoli. (2002). Fourth Dimensional Lie Symmetry Algebra and fourth order Ordinary Differential Equation. *Journal of Mathematical Physics*, 9, 24-35.

25. Oduor, M. O. (2005). Lie symmetry solutions of the generalised Burgers equation. PhD thesis, Maseno university, Kenya.

26. Olaniyi, S. (2013). A Numerical Study of Ito Equation and Sawada-Kotera Equation both of Time Fractional Type. *Advance Mathematics and ScienceJournal*, 2, 71-79.

27. Oliveri, F. (2010). Lie Symmetries of Differential Equations: Classical Results and Recent Contribution.658-706.

28. Olver, P. (1986). Applications of Lie Groups to Differential Equations. Springer:New York, NY,USA.

29. Omolo Ongati, N. (1997). Stability of Lie Groups of Non-linear Hyperbolic Equations. PhD thesis, University of Pretoria, South Africa.

30. Popovych, R. (2008). Reduction Operators of Linear Second Order Parabolic Equation. *Journal of Physics and Applied Mathematics* 41, 185-202.

31. Roman, C., and John, R. (2015). Lie and Conditioal Symmetries of a Class of Nonlinear (1+2)-Dimensional Boundary Value Problems. Symmetry, 7,1410-1435.

32. Roman, C., and Maksym, D. (2017). A (1+2)- Dimensional Simplified Keller-Segel Model: Lie Symmetry and Exact Solutions. Symmetry, 9, 13.

33. Roman, C., Mykola, S., and Oleksii, P. (2018). Lie and Q-Conditional Symmetries of Reaction Diffusion-Convection Equations with Exponential Nonlinearities and Their Application for Finding Exact Solutions. Symmetry, 10, 123.

34. Sakkaravarthi, K., Johnpillai, A., Durga, D., Kanna, T., and Lakshmanan, M. (2018). Lie Symmetry Analysis and Group Invariant Solutions of the nonlinear Helmholtz Equation. 1803-01622, 1-18.

35. Sh. Sadigh, B. (2012). Numerical Solution of Sawada-Kotera equation by using Iterative Methods. *International Journal of Industrial Mathematics*, 4, 269-288.

36. Vladimir, P., Ivan, U., and Edy, C. (2008). Finding lie symmetries of partial differential equations with MATHEMATICA :Applications to nonlinear fibre optics. softex, 280-291.

37. W.L., C., and Yu-Kun, Z. (1989). Backlund transformation for the Caudrey-Dodd-Gibbon-Sawada-Kotera Equation and its modified equation. *Journal of Mathematical Physics*, 30, 2065.

38. Xiazhi, H., Yinping, L., Xiaoyan, T., and Zhibin, L. (2016). Nonlocal symmetries and interaction solutions of the Sawada-Kotera equation. Modern Physics Letters B, 30.

39. Yaglom, I., Felix, K., and Sophus, L. (1988). Evolution of the idea of symmetry in the nineteenth century. Birkhauser, Boston, MA,USA.

40. Youwei, Z. (2015). Lie Symmetry Analysis to General Time-Fractional Korteweg-De Vries Equations. Fractional Differential Calculus, 5, 125-135.

41. Zablon, M., and Sogomo, K. (2007). Some hyndrons generalizations to starrett's method of solving first order ODEs by lie groups. Concept paper, Egerton University, Kenya.

42. Zhen-Yun, Q. (2018). On Periodic Wave Solution and Asymptotic Property of KdV-Sawada-Kotera Equation. Journal of the Physical Society of Japan, 76.

APPENDICES

Appendix I: UoK graduate school clearance

Appendix II: NACOSTI research authorization and permit

Appendix III: Originality report

Appendix IV: Publication document