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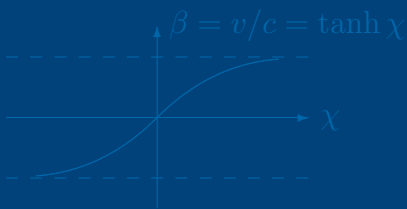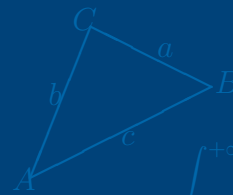
On the Structure theory of Finite Rings   View project

$\beta = v/c = \tanh \chi$

χ

$\int_{-\infty}^{+\infty} \frac{\sin t}{t} dt = \frac{\pi}{2}$

C

a

B

b

c

A

$\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$

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Effect of CD4$^+$ T-Cells and CD8$^+$ T-Cells on Psoriasis: A mathematical study

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Abstract

Psoriasis is a common chronic inflammatory skin disease that is differentiated by repeated occurrences of raised scaly and red skin plaques. It is generated through several applications of drugs, strains, physical wounds to the skin and also for infectivity. Psoriasis is identified by composite interactions of T-Cells, Dendritic Cells, Cytokines and downstream transcription factors (type 1 Cytokines network). The effects of T-Cells in dermal layer (CD4$^+$ T-Cells) are well-studied in the disease dynamics of Psoriasis from mathematical as well as biological context. But the concept of T-Cells in epidermal layer (CD8$^+$ T-Cells) for disease progression of Psoriasis has not yet been explored till now from mathematical avenue. Here we introduce both CD4$^+$ and CD8$^+$ T-Cell, Dendritic Cell and Keratinocyte population to notice the impact of them on immunopathogenic cell-biological mechanism of Psoriasis. Numerical simulation is also furnished to establish the analytical outcomes.

Effect of CD4$^+$ T-Cells and CD8$^+$ T-Cells on Psoriasis: A mathematical study

Abhirup Datta, Dipak Kumar Kesh and Priti Kumar Roy

Abstract. Psoriasis is a common chronic inflammatory skin disease that is differentiated by repeated occurrences of raised scaly and red skin plaques. It is generated through several applications of drugs, strains, physical wounds to the skin and also for infectivity. Psoriasis is identified by composite interactions of T-Cells, Dendritic Cells, Cytokines and downstream transcription factors (type 1 Cytokines network). The effects of T-Cells in dermal layer (CD4$^+$ T-Cells) are well-studied in the disease dynamics of Psoriasis from mathematical as well as biological context. But the concept of T-Cells in epidermal layer (CD8$^+$ T-Cells) for disease progression of Psoriasis has not yet been explored till now from mathematical avenue. Here we introduce both CD4$^+$ and CD8$^+$ T-Cell, Dendritic Cell and Keratinocyte population to notice the impact of them on immunopathogenic cell-biological mechanism of Psoriasis. Numerical simulation is also furnished to establish the analytical outcomes.

1. Introduction

A number of histological transformations can be noticed for increasing of lesions for Psoriasis: (1) a condensed epidermis (acanthosis) occurring from immediate proliferation of Keratinocyte, (2) for the unusual separation of Keratinocytes, granular stratum is decreased or not present (hypogranulosis) and preservation of nuclei by corneocytes is occurred (parakeratosis), (3) in the papillary dermis, noticeable dilation of blood vessels reasoning observable erythema and last of all (4) a thick inflammatory penetration, composed of groups for CD4$^+$ T helper Cells and DCs in the dermis and CD8$^+$ T-Cells and neutrophils in the epidermis [1]. In autocrine or paracrine mode, Cytokines usually intervene connections between cells that are in close immediacy by means of involvement for explicit receptors performance. Cytokines are normally capable to manipulate the proliferation, differentiation or discharge of pro-inflammatory or anti-inflammatory aspects by resident and employed cells [2]. IL-22 is almost certainly and extremely over expressed for up regulated IL-23 and IL-6 stages in Psoriasis [3], [4]. The capability of Cytokine IFN-γ is to capture the refined Keratinocytes proliferation intensely and unpredictably. It is done by controlling of recombinant human IFN-γ to the Psoriasis patient systemically [5]. TNF-α fabricating cells are assumed to persuade the adjoining endothelial cells

to manufacture bond molecules such as endothelial cell leukocyte bond molecule 1, vascular cell bond molecule 1 and intercellular bond molecule. It is furnished to speedily endorse the staffing of leukocytes into the skin and also to force epidermal Keratinocytes to produce chemotactic polypeptides such as IL-8. Leukocytes via intercellular adhesion molecule 1 are also kept [6]. Neutrophil attracting CXC chemokines are created by Keratinocytes, which are motivated by means of a potent pro-inflammatory Cytokine IL-17 [7], [8]. IL-17 may stimulate fibroblasts to generate Cytokine IL-6, which assigns naive T-Cells to Th 17. A positive feedback loop is activated potentially that enables Th17 inflammation [9]. Inside the epidermis, CD8$^+$ T-Cells constructing IL-17 have been recognized [10].

Roy and Bhadra prepared the basic mathematical model of the skin disease Psoriasis [11]. They also analyzed a comparative study for suppression on T-Cells and DCs individually and attained the better outcomes for suppression made on DCs [12]. Next, Roy and Datta expanded the mathematical model of Psoriasis introducing the half-saturation constant [13] and negative feedback control approach in delay induced system [14]. They also examined the effect of Cytokines network in the cell-biological structure of Psoriasis [15] and also the presence of CD8$^+$ T-Cells [16] was incorporated by Roy et al. in this disease dynamics of Psoriasis. Interaction between T-Cells and DCs and also interaction between T-Cells and Keratinocytes help to form Keratinocytes, whose excess production generates Psoriasis. From mathematical point of view, the research works regarding the cell-biological system of the disease Psoriasis were done related to the growth of Psoriasis but not on the persistence of Psoriasis. In our present research article, we generate the new concept of survival of the disease. The impact of T-Cells in dermal layer generally Th 1 CD4$^+$ T-Cells was considered in all the previous research articles on mathematical approaches for Psoriasis. The notion of epidermal T-Cells essentially CD8$^+$ T-Cells in cell-biology of Psoriasis has not yet been properly investigated till now from mathematical aspect. For that reason, we introduce four type of cell populations CD4$^+$ T-Cell, DC, CD8$^+$ T-Cell and Keratinocyte. Mainly, CD4$^+$ T-Cells are one of the factors for the growth of Psoriasis. But CD8$^+$ T-Cells in the epidermal layer are responsible for the persistence of the disease. At first, there is a interaction between CD4$^+$ T-Cells and DCs in the dermis. Another interaction of CD4$^+$ T-Cells with Keratinocytes occurs in the dermal phase. Next with help of various Cytokines network, the effects of the interactions come into the epidermis layer. Finally, Keratinocytes produced in the epidermis stage are one of the major causes to generate and persist the chronic disease Psoriasis. In this research article, we are interested to notice the effect of CD8$^+$ T-Cells along with CD4$^+$ T-Cells in the growth as well as consistence of the disease Psoriasis.

2. The Basic Assumptions and the Formulation of the Mathematical Model

We consider the mathematical model of chronic skin disease Psoriasis, where there are four different types of cell population, introduced into the model system: $l_D(t)$ is T-Cell density in dermal layer (mainly Th 1 CD4$^+$ T-Cells), $m(t)$ is Dendritic Cell density, $l_E(t)$ is epidermal T-Cell (mainly CD8$^+$ T-Cells) density and $k(t)$ is Keratinocyte density at time t. Here the constant rates of accumulation of T-Cells and Dendritic Cells are assumed by a and b respectively. Also the accumulation rate of resident T-Cells in secondary lymphatic organs is denoted by c. Again, ξ_1 is the activation rate of CD4$^+$ T-Cells by DCs and ξ_2 is the rate of activation of DCs by CD4$^+$ T-Cells. Further, the activation rate of Keratinocytes by CD4$^+$ T-Cells due to T-Cells mediated Cytokines is considered as η_1 and also β_1 is the rate of activation of CD8$^+$ T-Cells by DCs. Here the rate of migration of dermal layer T-Cells to epidermis is regarded as δ under the action of IL-6 and IFN-γ released by T-Cells themselves, which cause further stimulation of DCs and also β_2 is the rate of proliferation of epidermal T-Cells. Moreover, α_1 is

the rate of proliferation of epidermal T-Cells by activated Keratinocytes and the rate of activation of epidermal T-Cells by Keratinocytes under the influence of IL-8 is assumed as α_2. Here η_2 is the rate of further immigration of dermal T-Cells to epidermis guided by Keratinocytes and mediated by IL-8 and IL-20. Furthermore, λ_1 is the rate of proliferation of Keratinocytes mediated by Cytokines released from epidermal T-Cells like IL-17, IL-20 and also the rate of proliferation for Keratinocytes mediated by Cytokines released from Keratinocytes themselves is considered as λ_2. Finally, the per capita removal rates of CD4$^+$ T-Cells and CD8$^+$ T-Cells are denoted by μ_1 and μ_3 respectively and further μ_2 and μ_4 are the per capita removal rates of Dendritic Cell and Keratinocyte population. All the above mentioned parameters are always positive.

Assembling together the above assumptions, we may formulate the mathematical model of the disease Psoriasis given below:

$$
\begin{aligned}
\frac{dl_D}{dt} &= a - \xi_1 l_D m - \eta_1 l_D k - \mu_1 l_D, \\
\frac{dm}{dt} &= b - \xi_2 l_D m - \beta_1 l_E m - \mu_2 m, \\
\frac{dl_E}{dt} &= c + \delta l_D m + \beta_2 l_E m + \alpha_1 l_E + \eta_2 l_D k - \mu_3 l_E, \\
\frac{dk}{dt} &= \lambda_1 l_D k + \lambda_2 l_E k - \alpha_2 l_E - \mu_4 k,
\end{aligned}
\tag{1}
$$

where $l_D(0) > 0$, $m(0) > 0$, $l_E(0) > 0$ and $k(0) > 0$ at a specific time period t.

3. Theoretical Analysis of the Dynamical System

3.1. Existence, Uniqueness and Boundedness of the System

The right hand sides of system of equations (1) are smooth functions of the variables l_D, m, l_E and k and also the parameters, given that these quantities are always non-negative. Henceforth, the local existence, uniqueness and boundedness of the system dynamics are guaranteed in the positive octant. In the following theorem, we will exemplify that the linear combination of CD4$^+$ T-Cells, Dendritic Cells, CD8$^+$ T-Cells and Keratinocytes concentrations is less than a pre-assumed quantity. Alternatively, we can conclude that the solution of the dynamical system is bounded.

THEOREM 3.1. *The solution $x(t)$ of the system (1), where $x = (l_D, m, l_E, k)$, is uniformly bounded for $x_0 \in R_{0,+}^4$.*

Proof. We consider a function $U(t) : R_{0,+} \to R_{0,+}$ by $U(t) = l_D + m + l_E + k$. We study that U is precise and differentiable function on some maximal interval $(0, t_f)$. The time derivative of system of equations (1) is

$$
\begin{aligned}
\frac{dU}{dt} &= (a + b + c) + l_D m(\delta - \xi_1 - \xi_2) + l_D k(\eta_2 + \lambda_1 - \eta_1) + l_E m(\beta_2 - \beta_1) \\
&\quad + \lambda_2 l_E k + l_E(\alpha_1 - \alpha_2) - \mu_1 l_D - \mu_2 m - \mu_3 l_E - \mu_4 k.
\end{aligned}
$$

For the simplicity of computation, we here consider the sum of the rate of immigration of dermal T-Cells to epidermis guided by Keratinocytes and the rate of proliferation of Keratinocytes mediated by Cytokines released from epidermal T-Cells is almost same with the activation rate of Keratinocytes by CD4$^+$ T-Cells due to T-Cells mediated Cytokines, i.e., $\eta_2 + \lambda_1 = \eta_1$. Again we assume the rate of activation of CD8$^+$ T-Cells by DCs and the rate of proliferation of epidermal T-Cells is almost identical, i.e., $\beta_1 = \beta_2$. The rate of proliferation of epidermal

T-Cells by activated Keratinocytes and the rate of activation of epidermal T-Cells by Keratinocytes under the influence of IL-8 is assumed to be same for the sake of simplicity in the calculation ($\alpha_1 = \alpha_2$). Finally, we neglect the rate of proliferation for Keratinocytes mediated by Cytokines released from Keratinocytes themselves (λ_2) as we consider the rate of proliferation of Keratinocytes mediated by Cytokines released from epidermal T-Cells like IL-17, IL-20.

Therefore the above equation becomes,

$$\frac{dU}{dt} = (a + b + c) - l_D m(\xi_1 + \xi_2 - \delta) - \mu_1 l_D - \mu_2 m - \mu_3 l_E - \mu_4 k.$$

Now for all $\omega > 0$, the following inequality holds [12],

$$\frac{dU}{dt} + \omega U(t) \leq (a + b + c) - (\xi_1 + \xi_2 - \delta)(\frac{l_D^2 + m^2}{2}) - (\mu_1 - \omega)l_D - (\mu_2 - \omega)m - (\mu_3 - \omega)l_E$$

$$-(\mu_4 - \omega)k \leq (a + b + c) + \frac{(\mu_1 - \omega)^2 + (\mu_2 - \omega)^2 + (\mu_3 - \omega)^2}{2(\xi_1 + \xi_2 - \delta)} - (\mu_4 - \omega)k.$$

If we assume that $0 < \omega < \mu_4$, then there exists $\epsilon > 0$ such that

$$\frac{dU(t)}{dt} + \omega U(t) \leq \epsilon \text{ for each } t \in (0, t_f).$$

Let G(t,x)=$\epsilon - \omega x$, which ensures the Lipschtiz condition everywhere. Certainly,

$$\frac{dU(t)}{dt} \leq \epsilon - \omega U(t) = G(t, U(t)) \forall t \in (0, t_f).$$

Let $\frac{dy}{dt} = G(t, y) = \epsilon - \omega y$ and $y(0) = U(0) = U_0$. Now this ODE has the solution

$$y(t) = \frac{\epsilon}{\omega}(1 - e^{-\omega t}) + U_0 e^{-\omega t}.$$

It is obvious that $y(t)$ is bounded on $(0, t_f)$. By Comparison Theorem of Birkhoff and Rota (1982) [17],

$$U(t) \leq y(t) = \frac{\epsilon}{\omega}(1 - e^{-\omega t}) + U_0 e^{-\omega t} \,\forall\, t \in (0, t_f).$$

Now assume $t_f < \infty$, then $U(t_f) \leq y(t_f) < \infty$. Now the solution is established to be unique in some interval $(0, t_f)$ by the Picard-Lindelof Theorem. This contradicts our earlier assumption that $t_f < \infty$. Hence $U(t)$ must be bounded for all non-negative t and as a result $x(t)$ is uniformly bounded on $R_{0,+}$ [12].

∎

3.2. Permanence of the System

The system of equations (1) is considered to be permanent [18], if there exists a compact set D in the interior of $R_+^4 = \{(l_D(t), m(t), l_E(t), k(t)) \in R_+^4 \mid l_D(t) > 0, m(t) > 0, l_E(t) > 0, k(t) > 0\}$ such that, all solutions inside the interior of R_+^4 finally come into D and stay in D.

To study the permanence of the system of equations (1), let us consider $R_+^4 = \{(l_D(t), m(t), l_E(t), k(t)) \in R_+^4 \mid l_D(t) > 0, m(t) > 0, l_E(t) > 0, k(t) > 0\}$ is the positively invariant set of the system of equations (1) and $(l_D(t), m(t), l_E(t), k(t))$ is the arbitrary positive solution of the system of equations (1) with the help of positive initial value.

THEOREM 3.2. *For the system of equations (1) satisfying the initial condition $(l_D(0), m(0), l_E(0), k(0)) \in R_+^4$, there exists positive $l_{D\ max}^*, m_{max}^*, l_{E\ max}^*$ and k_{max}^*, such that for any $(l_D(t), m(t), l_E(t), k(t)) \in R_+^4$, $l_D(t) \leq l_{D\ max}^*, m(t) \leq m_{max}^*, l_E(t) \leq l_{E\ max}^*$ and $k(t) \leq k_{max}^*$ for large t.*

Proof. We have $U(t) = l_D + m + l_E + k$ is bounded, which is ensured for every non-negative t and consequently $x(t) = (l_D(t), m(t), l_E(t), k(t))$ is uniformly bounded on R_+. Hence the theorem.

∎

THEOREM 3.3. *For the system of equations (1) satisfying the initial condition $(l_D(0), m(0), l_E(0),$ $k(0)) \in R_+^4$, there exists positive $l_{D\ min}^*$, m_{min}^*, $l_{E\ min}^*$ and k_{min}^*, such that for any $(l_D(t), m(t),$ $l_E(t), k(t)) \in R_+^4$, $l_D(t) \geq l_{D\ min}^*$, $m(t) \geq m_{min}^*$, $l_E(t) \geq l_{E\ min}^*$ and $k(t) \geq k_{min}^*$ for large t.*

Obviously the system is bounded below. So we achieve a compact set $D = \{l_D(t), m(t), l_E(t),$ $k(t) \mid l_{D\ min}^* \leq l_D(t) \leq l_{D\ max}^*$, $m_{min}^* \leq m(t) \leq m_{max}^*$, $l_{E\ min}^* \leq l_E(t) \leq l_{E\ max}^*$ and $k_{min}^* \leq$ $k(t) \leq k_{max}^*\}$ corresponding to the system of equations (1), where each and every solution of the system with positive initial value will enter in to the compact set D and stay in D. Hence the positive invariant solution of the system of equations (1) is permanent [12].

3.3. Total Cell Count

Now, if assume the relation $\mu_1 < \mu_2 < \mu_3 < \mu_4$ then,

$$\frac{dT_{tot}}{dt} \equiv \frac{d(l_D + m + l_E + k)}{dt} \leq (a + b + c) - \mu_1(l_D + m + l_E + k),$$

providing the sum of the activation rate of CD4$^+$ T-Cells by DCs and the rate of activation of DCs by CD4$^+$ T-Cells is the same with the rate of migration of dermal layer T-Cells to epidermis. Furthermore, we here consider the sum of the rate of immigration of dermal T-Cells to epidermis guided by Keratinocytes and the rate of proliferation of Keratinocytes mediated by Cytokines released from epidermal T-Cells is almost same with the activation rate of Keratinocytes by CD4$^+$ T-Cells due to T-Cells mediated Cytokines. Again we assume the rate of activation of CD8$^+$ T-Cells by DCs and the rate of proliferation of epidermal T-Cells is almost identical. The rate of proliferation of epidermal T-Cells by activated Keratinocytes and the rate of activation of epidermal T-Cells by Keratinocytes under the influence of IL-8 is assumed to be same for the sake of simplicity in the calculation. Finally, the rate of proliferation for Keratinocytes mediated by Cytokines released from Keratinocytes themselves is neglected because we consider the rate of proliferation of Keratinocytes mediated by Cytokines released from epidermal T-Cells like IL-17, IL-20.

Lemma 3.4. *Consider v is a variable satisfying $v'(t) < d - f(\phi)v(t)$, where d is a constant and $f(\phi)$ is independent of v and t. Then If $v(0) < \frac{d}{f(\phi)}$, it pursues that $v(t) < \frac{d}{f(\phi)}$ for every t.*

Proof. See Smith and Wahl (2004, Lemma 4.1) [19].

∎

Remark 3.5. *If the inequalities are reversed, Lemma 3.4 also holds.*

Applying the above Lemma 3.4, we can state that $T_{tot} < \frac{a+b+c}{\mu_1}$, if $T_{tot}(0) < \frac{a+b+c}{\mu_1}$. Therefore, if the above mentioned assumptions are hold, then the limiting value of the total cell population should not exceed the quantity $\frac{a+b+c}{\mu_1}$ [20].

3.4. Equilibria of the System

The system of equations (1) has only the interior equilibrium point $E^*(l_D^*, m^*, l_E^*, k^*)$. From the first equation of the system (1), we have $l_D^* = \frac{a}{\xi_1 m^* + \eta_1 k^* + \mu_1}$, which is always positive. From the second equation we find that $m^* = \frac{b}{\xi_2 l_D^* + \beta_1 l_E^* + \mu_2}$, which is again positive by our assumptions.

FIGURE 1. Population densities of CD4$^+$ T-Cell, Dendritic Cell, CD8$^+$ T-Cell and Keratinocyte, which are plotted as a function of time and the value of the parameters are given in Table.

FIGURE 2. Population densities of CD4$^+$ T-Cell, Dendritic Cell, CD8$^+$ T-Cell and Keratinocyte, which are plotted as a function of time, β_1=0.0005 and β_2=0.0003 and the value of the other parameters are given in Table.

we get $l_E^* = \frac{c + l_D^*(\delta m^* + \eta_2 k^*)}{\mu_3 - (\beta_2 m^* + \alpha_1)}$, which is positive if $\mu_3 > \beta_2 m^* + \alpha_1$ from the third equation of the system (1). Finally from the fourth as well as last equation of the model system we obtain,

$k^* = \frac{\alpha_2 l_E^*}{\lambda_1 l_D^* + \lambda_2 l_E^* - \mu_4}$, which is positive when $\lambda_1 l_D^* + \lambda_2 l_E^* > \mu_4$.

Now the variational matrix at the interior equilibrium point $E^*(l_D^*, m^*, l_E^*, k^*)$ is given by $V(l_D^*, m^*, l_E^*, k^*) =$

$$
\begin{pmatrix}
-\xi_1 m^* - \eta_1 k^* - \mu_1 & -\xi_1 l_D^* & 0 & -\eta_1 l_D^* \\
-\xi_2 m^* & -\xi_2 l_D^* - \beta_1 l_E^* - \mu_2 & -\beta_1 m^* & 0 \\
\delta m^* + \eta_2 k^* & \delta l_D^* + \beta_2 l_E^* & \beta_2 m^* + \alpha_1 - \mu_3 & \eta_2 l_D^* \\
\lambda_1 k^* & 0 & \lambda_2 k^* - \alpha_2 & \lambda_1 l_D^* + \lambda_2 l_E^* - \mu_4
\end{pmatrix},
$$

i.e.,

$$
V(l_D^*, m^*, l_E^*, k^*) =
\begin{pmatrix}
-\frac{a}{l_D^*} & -\xi_1 l_D^* & 0 & -\eta_1 l_D^* \\
-\xi_2 m^* & -\frac{b}{m^*} & -\beta_1 m^* & 0 \\
\delta m^* + \eta_2 k^* & \delta l_D^* + \beta_2 l_E^* & \beta_2 m^* + \alpha_1 - \mu_3 & \eta_2 l_D^* \\
\lambda_1 k^* & 0 & \lambda_2 k^* - \alpha_2 & \frac{\alpha_2 l_E^*}{k^*}
\end{pmatrix},
$$

i.e.,

$$
V(l_D^*, m^*, l_E^*, k^*) =
\begin{pmatrix}
-\frac{a}{l_D^*} & -\xi_1 l_D^* & 0 & -\eta_1 l_D^* \\
-\xi_2 m^* & -\frac{b}{m^*} & -\beta_1 m^* & 0 \\
d_{31} & d_{32} & d_{33} & \eta_2 l_D^* \\
\lambda_1 k^* & 0 & d_{43} & \frac{\alpha_2 l_E^*}{k^*}
\end{pmatrix},
$$

where $d_{31} = \delta m^* + \eta_2 k^*$, $d_{32} = \delta l_D^* + \beta_2 l_E^*$, $d_{33} = \beta_2 m^* + \alpha_1 - \mu_3$ and $d_{43} = \lambda_2 k^* - \alpha_2$.

The characteristic equation is given by,

$$\phi^4 + A_1 \phi^3 + A_2 \phi^2 + A_3 \phi + A_4 = 0, \tag{2}$$

where

$$A_1 = -d_{33} + \frac{a}{l_D^*} + \frac{b}{m^*} - \frac{\alpha_2 l_E^*}{k^*},$$

$$
A_2 = -\frac{a(d_{33})}{l_D^*} - \frac{b(d_{33})}{m^*} + \frac{ab}{l_D^* m^*} + \frac{\alpha_2(d_{33})l_E^*}{k^*} - \frac{a\alpha_2 l_E^*}{l_D^* k^*} - \frac{b\alpha_2 l_E^*}{m^* k^*}
$$
$$
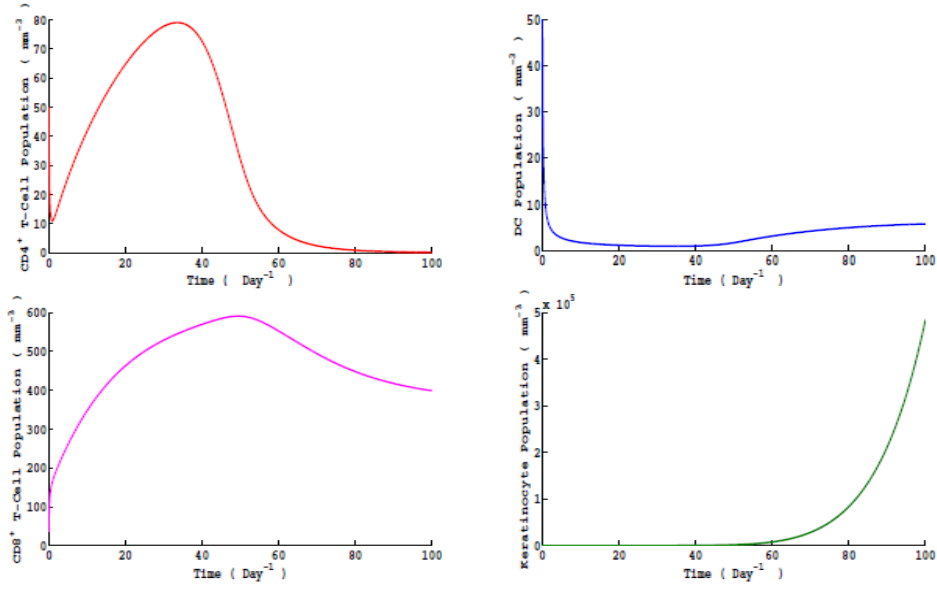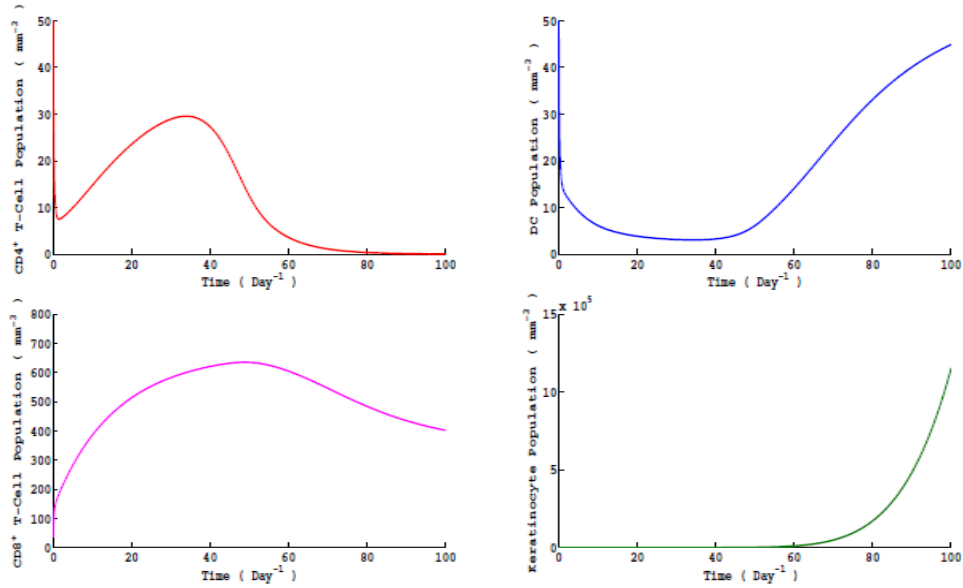+ \beta_1(d_{32})m^* + \eta_1 \lambda_1 l_D^* k^* - \eta_2(d_{43})l_D^* - \xi_1 \xi_2 l_D^* m^*,
$$
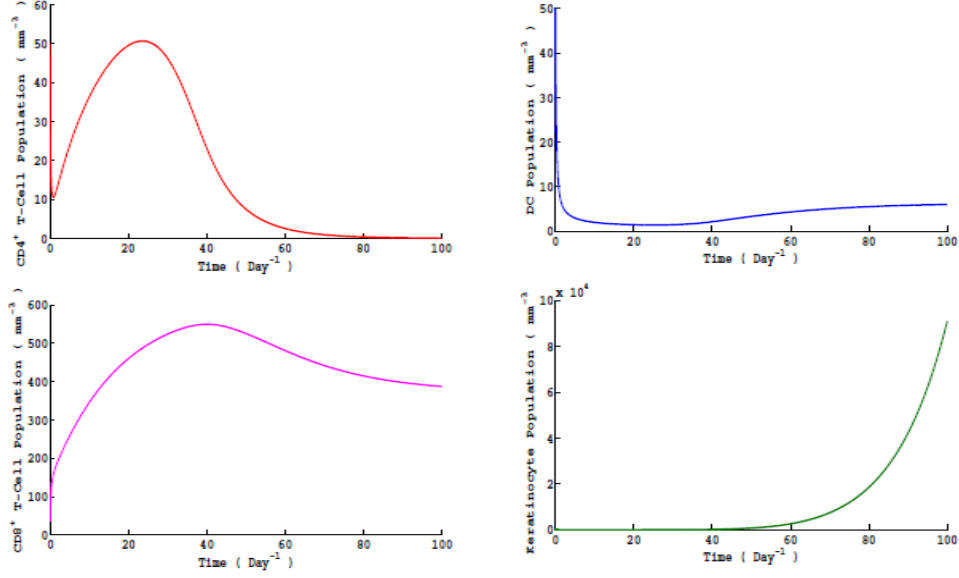
FIGURE 3. Population densities of CD4$^+$ T-Cell, Dendritic Cell, CD8$^+$ T-Cell and Keratinocyte, which are plotted as a function of time η_1=0.002 and η_2=0.001 and the value of the other parameters are given in Table.

$$
\begin{aligned}
A_3 \;=\; & -\frac{ab(d_{33})}{l_D^* m^*} + \frac{a(d_{33})\alpha_2 l_E^*}{l_D^* k^*} + \frac{b(d_{33})\alpha_2 l_E^*}{m^* k^*} - \frac{ab\alpha_2 l_E^*}{l_D^* m^* k^*} + \frac{a(d_{32})\beta_1 m^*}{l_D^*} - \frac{\alpha_2 \beta_1 (d_{32}) l_E^* m^*}{k^*} \\
& + \eta_1(d_{31})(d_{43})l_D^* - \eta_1 \lambda_1 (d_{33})l_D^* k^* + \frac{b\eta_1 \lambda_1 l_D^* k^*}{m^*} - a\eta_2(d_{43}) - \frac{b\eta_2(d_{43})l_D^*}{m^*} \\
& - \beta_1 \xi_1 (d_{31})l_D^* m^* + \xi_1 \xi_2 (d_{33})l_D^* m^* + \frac{\alpha_2 \xi_1 \xi_2 l_D^* l_E^* m^*}{k^*},
\end{aligned}
$$

and

$$
\begin{aligned}
A_4 \;=\; & \frac{ab(d_{33})\alpha_2 l_E^*}{l_D^* m^* k^*} - \frac{a(d_{32})\alpha_2 \beta_1 l_E^* m^*}{l_D^* k^*} + \frac{b\eta_1(d_{31})(d_{43})l_D^*}{m^*} - \xi_2 \eta_1 (d_{32})(d_{43})l_D^* m^* \\
& - \frac{b(d_{33})\eta_1 \lambda_1 l_D^* k^*}{m^*} + \beta_1 \eta_1 \lambda_1 (d_{32})l_D^* m^* k^* - \frac{ab\eta_2(d_{43})}{m^*} + \frac{\alpha_2 \beta_1 \xi_1 (d_{31})l_D^* l_E^* m^*}{k^*} \\
& - \frac{\alpha_2 \xi_1 \xi_2 (d_{33})l_D^* l_E^* m^*}{k^*} + \eta_2 \xi_1 \xi_2 (d_{43})(l_D^*)^2 m^* - \beta_1 \lambda_1 \eta_2 \xi_1 (l_D^*)^2 m^* k^*.
\end{aligned}
$$

From Routh-Hurwitz criterion, the interior equilibrium point $E^*(l_D^*, m^*, l_E^*, k^*)$ is said to be stable if the following conditions are hold: (a) $A_1 > 0$, (b) $A_4 > 0$, (c) $A_1 A_2 - A_3 > 0$ and (d) $[(A_1 A_2 - A_3)A_3] - A_1^2 A_4 > 0$.

Now, according to the Routh-Hurwitz criterion, the interior equilibrium point E^* is stable if

(1) $\frac{a}{l_D^*} > d_{33}$, (2) $\frac{b}{m^*} > \frac{\alpha_2 l_E^*}{k^*}$, (3) $d_{31}d_{43} > \lambda_1 d_{33} k^*$, (4) $\beta_1 \lambda_1 k^* > \xi_2 d_{43}$, (5) $ak^* > \frac{d_{31}d_{43}l_D^*}{\lambda_1} + \alpha_2 l_D^* l_E^*$ and (6) $[(A_1 A_2 - A_3)A_3] - A_1^2 A_4 > 0$.

4. Numerical Simulation

In the preceding section, we have introduced analytical method for qualitative study of the system. In this division, we execute numerical simulation of the model system. We have taken approximate values of the parameters during our analytical outcomes. Numerical values of the model parameters, mainly taken from [12], have been specified in the **Table** given below.

Table. Values of parameters used for model system

Parameter	Default Values (Day $^{-1}$)	Parameter	Default Values (Day $^{-1}$)
a	15 mm^{-3}	δ	0.35 mm^3
b	12 mm^{-3}	α_1	0.007
c	9 mm^{-3}	α_2	0.004
ξ_1	0.15 mm^3	λ_1	0.0003 mm^3
ξ_2	0.12 mm^3	λ_2	0.0004 mm^3
η_1	0.0002 mm^3	μ_1	0.02
η_2	0.0001 mm^3	μ_2	0.05
β_1	0.005 mm^3	μ_3	0.07
β_2	0.003 mm^3	μ_4	0.08

We are trying to observe the cell behavioral outline of different types of cells, involved in our model system for deviation in the values of the model parameters. In **Figure 1**, CD4$^+$ T-Cell population increases from initial position to higher level and then decreases to ground level gradually in first **100** days. From very beginning DC population decreases and afterward slightly increases. CD8$^+$ T-Cell population also increases from initial situation up to the highest point and next tendency of decreasing nature is observed. Finally Keratinocyte population remains constant at first near about **60** days. Then it increases sharply to the pick within **100** days. In **Figure 2**, we decrease the values of the rate of activation of CD8$^+$ T-Cells by DCs (β_1) and the rate of proliferation of epidermal T-Cells (β_2). We consider here β_1=0.0005 and β_2=0.0003 for **Figure 2**. Now, CD4$^+$ T-Cell population increases but not as much as **Figure 1**. DC population after **40** days increases gradually. Same behaviors are observed for the case of CD8$^+$ T-Cell and Keratinocyte population as like **Figure 1**. In **Figure 3**, we increase the values of the activation rate of Keratinocytes by CD4$^+$ T-Cells due to T-Cells mediated Cytokines (η_1) and the rate of further immigration of dermal T-Cells to epidermis guided by Keratinocytes and mediated by IL-8 and IL-20 (η_2). We assume η_1=0.002 and η_2=0.001. Here CD4$^+$ T-Cell population increases but range lies between **Figure 1** and **Figure 2**. The nature of DC population behaves as like as **Figure 1**. The behaviors of the other two populations CD8$^+$ T-Cell and Keratinocyte population perform as like as the previous figures.

5. Discussion

In our article, we introduce CD4$^+$ T-Cell and CD8$^+$ T-Cell populations. Next, we obtain unique equilibrium point $E^*(l_D^*, m^*, l_E^*, k^*)$ (interior equilibrium point) and then attain its stability analysis. From Routh-Hurwitz condition, the interior equilibrium point $E^*(l_D^*, m^*, l_E^*, k^*)$ is said to be stable if the following conditions are hold: (1) $\frac{a}{l_D^*} > d_{33}$, (2) $\frac{b}{m^*} > \frac{\alpha_2 l_E^*}{k^*}$, (3) $d_{31}d_{43} > \lambda_1 d_{33}k^*$, (4) $\beta_1\lambda_1 k^* > \xi_2 d_{43}$, (5) $ak^* > \frac{d_{31}d_{43}l_D^*}{\lambda_1} + \alpha_2 l_D^* l_E^*$ and (6) $\Delta A_3 - A_1^2 A_4 > 0$, where $\Delta = A_1 A_2 - A_3 > 0$.

6. Conclusion

In the cell-biological scenario, we can conclude that if the values of the rate of activation of CD8$^+$ T-Cells by DCs (β_1) and the rate of proliferation of epidermal T-Cells (β_2) are decreased, then DC population is increased. As a result, CD4$^+$ T-Cell population is decreased and also CD8$^+$ T-Cell population is increased. Further the values of the activation rate of Keratinocytes

by CD4$^+$ T-Cells due to T-Cells mediated Cytokines (η_1) and the rate of further immigration of dermal T-Cells to epidermis guided by Keratinocytes and mediated by IL-8 and IL-20 (η_2) are increased. As an effect, CD4$^+$ T-Cell and CD8$^+$ T-Cell populations both are decreased. DC population is unchanged for the variation of η_1 and η_2. The deviations in the values of β_1, β_2, η_1 and η_2 have not any significant impact on Keratinocyte population. The increasing nature of Keratinocytes remains unchanged. Therefore, we may bring to an end that CD8$^+$ T-Cell along with CD4$^+$ T-Cell population has an effect not only growing criteria of the disease but also the survival of Psoriasis. Thus if we can communicate our research findings into the treatment policy of Psoriasis, we are able to expect a better outcome for welfare of our society.

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Mathematical Modelling of The East Africa Marine Systems (TEAMS) Fiber OPtic

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Abstract

The concept of high bandwidth capabilities and low attenuation characteristics make it ideal for gigabit data transmission possible because light energy can be modelled in a wave. Mathematics and communication plays an integral role in today's world economic platform especially in large scale transmission of data and voice. We consider a cylindrical dielectric waveguide made of silica glass. The discussion will be based on the nature and behaviour of some of the ordinary differential equations (ODE's) and the partial differential equations (PDE's) namely; Maxwell equations, Schrödinger's equations and the Bessel functions and their interactions and applications then investigate the fiber optics solutions theory in communication engineering which plays a vital role in transmission capacity than metallic cables and therefore suited to the increase demand for high transmission capacity and speed. The problem involves studying the motion of sound which is a wave subjected to a sinusoidal forcing function. In this case the focus will be on Kenya being one of the developing countries in communication to the rest of the East African countries: Uganda, Rwanda, Burundi and Tanzania through cross-border connectivity arrangements and how fiber cables have enabled this happen in sharing data as fast as possible. The differential equations used in describing pulse propagation in the dispersion-dominated nonlinear fiber channel should demonstrate an agreement between the analytical results and the numeric. This technique is aimed at simplification of digital signal processing of nonlinear impairments represented graphically.

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Abstract. The concept of high bandwidth capabilities and low attenuation characteristics make it ideal for gigabit data transmission possible because light energy can be modelled in a wave. Mathematics and communication plays an integral role in today's world economic platform especially in large scale transmission of data and voice. We consider a cylindrical dielectric waveguide made of silica glass. The discussion will be based on the nature and behaviour of some of the ordinary differential equations (ODE's) and the partial differential equations (PDE's) namely; Maxwell equations, Schrödinger's equations and the Bessel functions and their interactions and applications then investigate the fiber optics solutions theory in communication engineering which plays a vital role in transmission capacity than metallic cables and therefore suited to the increase demand for high transmission capacity and speed. The problem involves studying the motion of sound which is a wave subjected to a sinusoidal forcing function. In this case the focus will be on Kenya being one of the developing countries in communication to the rest of the East African countries: Uganda, Rwanda, Burundi and Tanzania through cross-border connectivity arrangements and how fiber cables have enabled this happen in sharing data as fast as possible. The differential equations used in describing pulse propagation in the dispersion-dominated nonlinear fiber channel should demonstrate an agreement between the analytical results and the numeric. This technique is aimed at simplification of digital signal processing of nonlinear impairments represented graphically.

Nomenclature

v	Propagation distance along the fiber
t	Propagation distance along the time
V_{gr}	Carrier group velocity
D	Dispersion coefficient.
μ	Envelope components for the real function of z
ν	Envelope components for the real function of τ
λ	Arbitrary parameter
F	Unknown function of t and its derivatives
$|A|^2$	Pulse amplitude
α	Fiber losses
β_2	Chromatic dispersion
γ	Fiber nonlinearity
n_2	Kerr nonlinear index coefficient
A_{eff}	Effective core area
c	Light velocity in vacuum
β_1	First order dispersion
k	Wave vector [radians/m] ν
ω	Angular frequency [radians/sec]
$\lambda 0$	Wavelength in vacuum [m]
n	Refractive index
$J_n(x)$	Frequency component of magnitude
Γ	Gamma function
n_1	Refractive index of the medium the light is leaving
θ_1	Incident angle between the light beam and the normal
n_2	Refractive index of the material the light is entering
θ_2	Refractive angle between the light ray and the normal
n	Side frequency number/order of differential equation
$x \ or \ l$	Modulation index
$D \ or \ \frac{dy}{dx}$	Differential operator
α	Phase difference
fm	Argument.

1. Introduction

The East African Marine System (TEAMS) is a 5,000-km fibre-optic undersea cable linking the Kenya's coastal town of Mombasa with Fujairah in the UAE was built at a cost of USD 130 million as a joint venture between the government of Kenya and Kenyan operators as follows; 42.5% – Telkom Kenya Ltd, 22.5% – Safaricom Ltd,10% – Kenya Data Networks Ltd, 10% – Econet/Essar Telecom Ltd, 5% – Wananchi Group, 3.75% – Jamii Telecom Ltd, 1.25% – Broadband Access/AccessKenya Ltd, 1.25% – Africa Fibrenet (Uganda) Ltd, 1.25% – InHand Ltd, 1.25% – iQuip Ltd, 1.25% – Flashcom Ltd.

According to Daily Nation, Thursday 30th December 2010.Construction of the cable began in January 2008 on the Emirates' side and arrived in the Kenyan port city of Mombasa on 12th June 2009. Cable construction was completed in August 2009 and the Teams cable went live for commercial service on 1st October 2009. TEAMS cable is connected to the Kenya national fiber backbone network and other major backhaul providers, thus extending the gigabit submarine capacity to the rest of the East African countries: Uganda, Rwanda, Burundi and Tanzania through cross-border connectivity arrangements.TEAMS is strategically positioned in the EA region to spur rapid social, economic and educational development by providing

reliable, worldwide gigabit connectivity to ICT operators, ISPs and large bandwidth users at cost-efficient and competitive rates.

A fiber optic consists of a central core in which light is guided, embedded in an outer cladding of slightly lower refractive index. Light rays incident on the core-cladding boundary angles greater than the critical angle undergo total internal reflection and are guided through the core without refraction.

Recent developments in fibre optics in communication system prove almost zero loss and infinite bandwidth. Indeed, optical fibre communication systems are fulfilling the increased demand on communication links, especially with the proliferation of the internet, Chynoweth (1976). Fiber cables have proven to provide faster and cheaper internet connectivity than the traditional satellites; be it streaming videos live or downloading high-definition videos in a very short time. As a result of recent technological advances in fabrication, light can be guided through fibre optic with very minimal loss and these optical fibers are replacing copper coaxial cables as the preferred transmission medium for electromagnetic waves. The low attenuation and superior signal quality of fibre optic communication systems allow communications signals to be transmitted over much longer distances than metallic-based systems without signal regeneration Ishigure *et al* (1996).
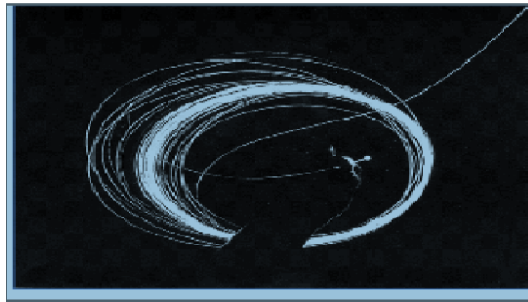
FIGURE 1. A long, thin optical fiber transmitting a light beam (Photograph courtesy Dr. Chynoweth (1976))

An optical fiber (Figure 1) consists of a central glass core of radius "a" surrounded by an outer cladding made of glass with a slightly lower refractive index. The corresponding refractive index distribution (in the transverse direction) is given by:

$n = n_1$ for $r < a$

$n = n_2$ for $r > a$

The core diameter $d = 2a$ of a typical telecommunication-grade multimode fiber is approximately 62.5 μm with an outer cladding diameter of 125 μm. The cladding index n_2 is approximately 1.45 (pure silica), and the core index n_1, barely larger, around 1.465μm. The cladding is usually pure silica while the core is usually silica doped with germanium, which increases the refractive index slightly from n_2 to n_1. The core and cladding are fused together during the manufacturing process and typically not separable. An outside plastic buffer is usually added to protect the fiber from environmental contaminants. The diagram below by STEP (2008)
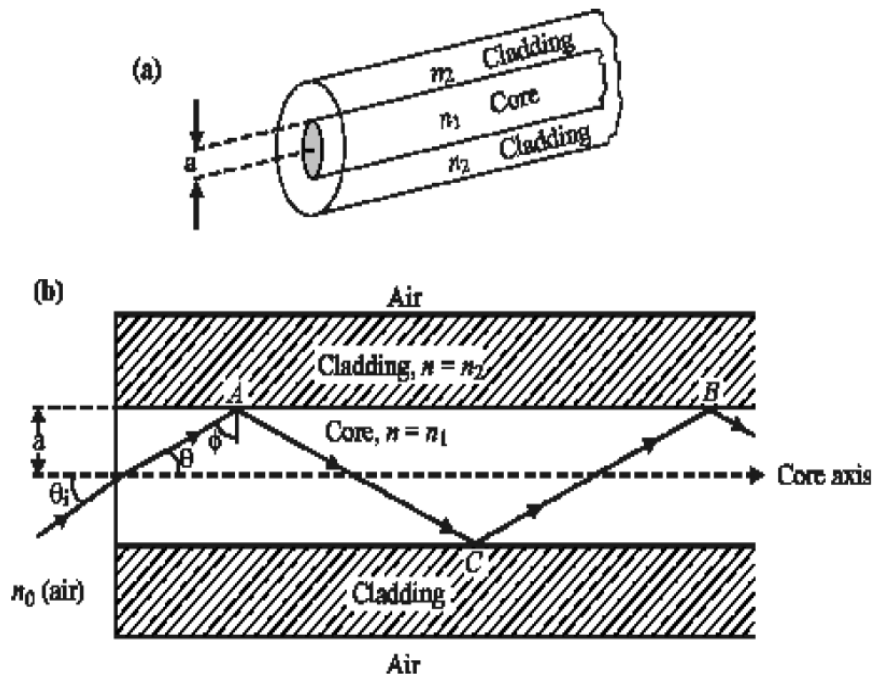
FIGURE 2. *(a) A glass fiber consists of a cylindrical central core surrounded by a cladding material of slightly lower refractive index. (b) Light rays impinging on the core-cladding interface at an angle φ greater than the critical angle φc are trapped inside the core of the fiber and reflected back and forth (A,C, B, etc.) along the core-cladding interface.*

2. Literature Review

Similar research has been done by Kundaeli (2001) analysed an optical fiber communication system using laser rate equations where the effect of electrical pulse shaping on dispersion include pulse distortion was investigated using computer simulation techniques. The results showed that the detrimental effects of the dispersion can be greatly reduced without incurring very high costs. More research has been done on dispersion-dominated nonlinear fiber-optic channels, Sergei *et al* (2012) used used a technique aimed at simplification of the following digital signal processing of nonlinear impairments by using a model describing pulse propagation in the dispersion-dominated nonlinear fiber channel. In the limit of very strong initial pre-dispersion the nonlinear propagation equations for each Fourier mode become local and decoupled.

Deqiang *et al* (2013) studied performance of concealed optical wireless communication link based on modulated retro-reflector. The simulation results show that wavelength, angle of incidence, and refractive index of CCR are the key factors influencing the performance of link, and the SNR fluctuation is decreased by reflecting of CCR.

Fang D. *et al* (2014) studied the design of a fully-fiber multi-chord interferometer and a new phase-shift demodulation method for field-reversed configuration where the noteworthy feature was mathematically compared to the two divided interference signals, which had the same phase-shift caused by the electron density but possess the different initial phase and low angular frequencies. It was possible to read the plasma density directly on the oscilloscope by the original mathematic demodulation method without a camera.

Buczynski *et al* (2014), who presented experimental and numerical progress in the development of ultrafast solitonic nonlinear directional couplers utilizing multi-component glass dual-core

photonic crystal fibers. Due to the fibre birefringence the switching wavelength can be tuned by rotating the polarization of the excitation field.The numerical studies were focused on single fundamental soliton switching that exhibits high extinction ratios. The resultant coupler design was further analysed from the aspect of nonlinear propagation based on coupled nonlinear Schrödinger equations. The simulated nonlinear directional coupler indicates the possibility to realize a high-extinction-ratio switch of sub-nJ 100 fs pulses which are simultaneously compressed below 15 fs.

We therefore fell there is a way we can summarize some important equations needed in the analysis of light propagations in fibers that lead to an increase speed in data transmission.

3. Mathematical Formulation

3.1. Bessel Differential Equations

Bessel differential equations given by;

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - n^2\right) y = 0, \tag{1}$$

where n is the order of differential equation and it is a given number, real or complex. The point $x = 0$ is a regular singularity, and is the Bessel functions which is a solution of equation (1) which has a solution of the form

$$y = \sum_{K=0}^{8} a_k x^{m+k} \tag{2}$$

Using power series and substituting in equation (2), we get the solution

$$J_n(x) = \frac{x^n}{2^n \Gamma(n+1)} \left\{1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2.4(2n+2)(2n+4)} - \ldots\right\}. \tag{3}$$

The Bessel functions $J_n(x)$ has power series that is convergent, with better convergence than the familiar series for the exponential or trigonometric functions which can also be expressed as the sum for integral values of n, Basmadjian (2002), where n-is a positive integer and not zero. It can be written as an infinite polynomial with terms derived from the gamma function, Γ.

$$J_n(x) = \sum_{K=0}^{8} \frac{(-1)^k \left(\frac{x}{2}\right)^{n+2k}}{k!\Gamma(n+k+1)} \tag{4}$$

We considered only the case where n is an integer. The canonical solutions considered are the Bessel functions of the first kind, $J_n(x)$ non-singular at $x = 0$.

n particular, putting $n = 1$, in the above equation Bessel function of order one is given by

$$J_1(x) = \frac{x}{2} - \frac{x^3}{2^4.4} + \frac{x^5}{2^2 4^2.6} - \ldots \tag{5}$$

The function $J_1(x)$ is oscillating with a decreasing amplitude and varying period. The roots of these functions are not completely regularly spaced and the amplitude of the wave decreases with the increase value of x, which looks similar to a sine function.

The amplitude of the carrier signal is a function of the modulation index and under some conditions; its amplitude can sometimes converge to zero. This does not mean that the signal disappears, but rather that all of the broadcast energy is redistributed to the side frequencies.
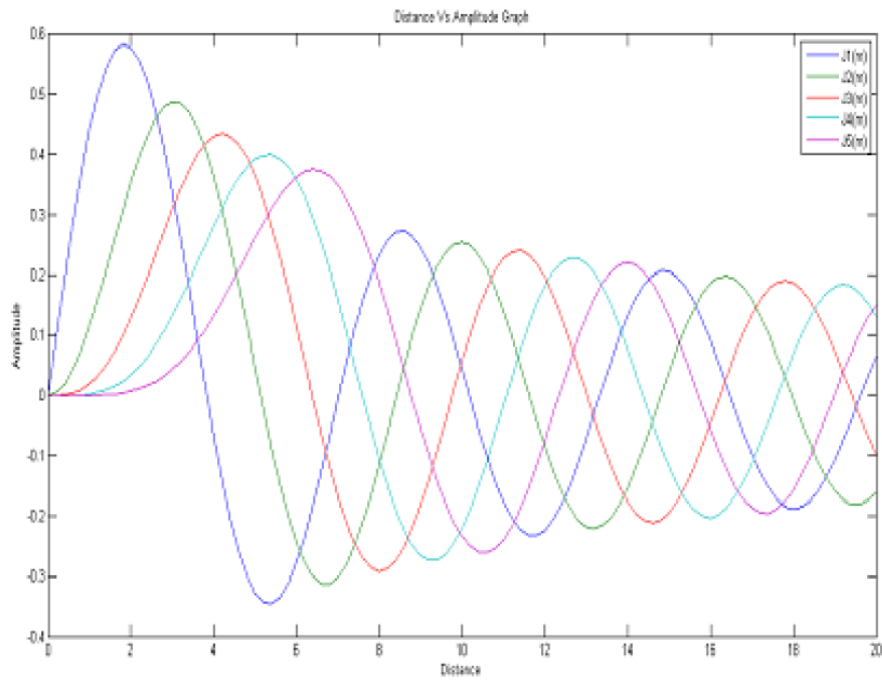
FIGURE 3. Plots of Bessel Functions of the first kind at $0 \leq fm \leq 20$

For computational purposes, we use smaller measures to illustrate the concepts in all the areas in this work.

For instance, if we increase the distance to 80 and 200 units, we obtain the image in Figure 4 and Figure 5.

FIGURE 4. Plots of Bessel Functions of the first kind at $0 \leq fm \leq 80$
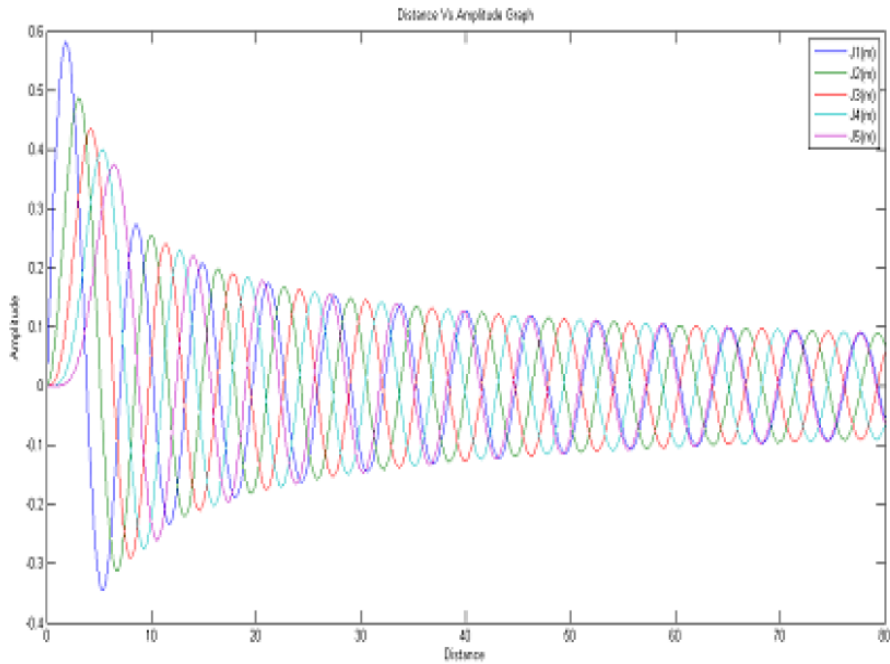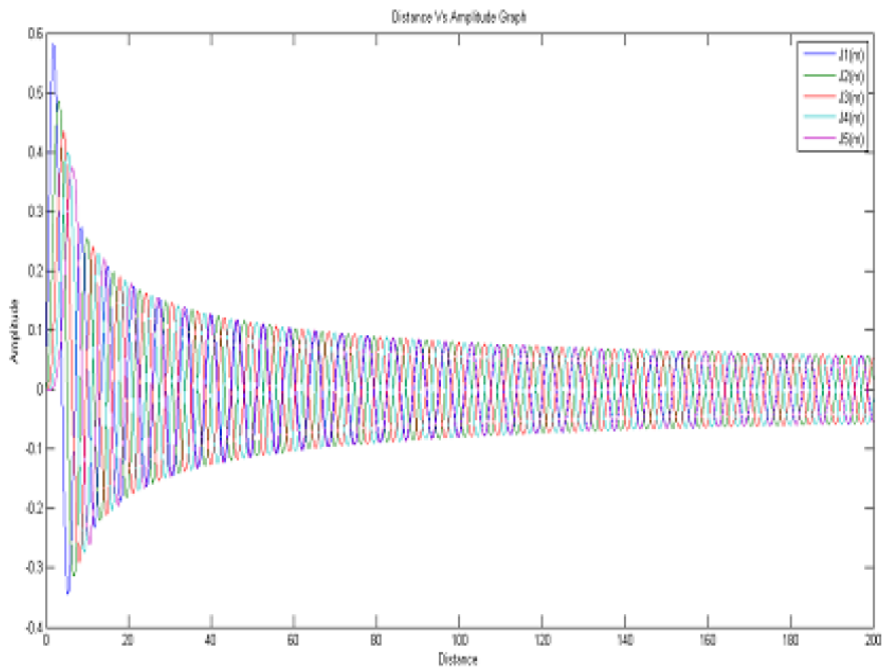

FIGURE 5. Plots of Bessel Functions of the first kind at $0 \leq fm \leq 200$

This modulated signal is consisting of five frequency components added together to give the appearance of a sine wave. In this case, the frequency is varying with time when

displayed in the distance domain. From the three graphs of Bessel Functions above, the values of the term $J_n\,(mf)$ which gives amplitude of n-th side band with modulation index mf are determined using series solution as mentioned in equation (2) and the values of the $J_n\,(mf)$ terms are calculated. Mathematically, the results of the numerical computation of the values of $J_0\,(mf)$, $J_1\,(mf)$, $J_2\,(mf)$, $J_3\,(mf)$, $J_4\,(mf)$ and $J_5\,(mf)$ are plotted. It can be observed from the graph that for small values of mf, the only Bessel functions with any significant amplitude are $J_0\,(mf)$ and $J_1\,(mf)$, while the amplitude of the higher-order ($n > 1$) sideband pairs is very small Saxena *et al* (2009). As mf increases, the amplitude of the rest frequency decreases and the amplitude of the higher-order sidebands increase, thus an increasing signal bandwidth. Therefore, the amplitudes of the higher-order sideband pairs eventually approach zero.

3.2. Schrödinger's Equation

Schrödinger's equation is named after Erwin Schröodinger, 1887-1961. This is a second order partial differential equation. These equations will be used where the transmission of multiple channels should address the impacts of dispersion and nonlinear phenomena that occur during transmission. The NLS equation for slowly varying amplitude $\Psi(z,\tau)$ is given by

$$i\Psi_z + \frac{1}{2}D\Psi_{\tau\tau} + \gamma|\Psi|^2\Psi = 0, \tag{6}$$

where $\tau = t - \frac{z}{V_{gr}}$ If we substitute $\Psi = \mu + i\upsilon$ equation (6), Malomed (2002), we get the following differential equation for μ and υ

$$-\nu_z + \frac{1}{2}D\mu_{\tau\tau} + \gamma\left(\mu^2 + \nu^2\right)\mu = 0 \tag{7}$$

for the real part of equation (6) and

$$\mu_z + \frac{1}{2}D\nu_{\tau\tau} + \gamma\left(\mu^2 + \nu^2\right)\nu = 0 \tag{8}$$

for the imaginary part of equation (6)

We construct the following trial functional in order to establish variation formulation, He (2004)

$$J\left(\mu,\upsilon\right) = \int \{\lambda\nu\mu_z - (1-\lambda)\,\nu_z\mu - \frac{1}{4}D\mu_\tau^2 + \frac{1}{2}\gamma\left(\mu^4 + 2\mu^2\nu^2\right) + F\}dzd\tau \tag{9}$$

The advantage of the above trial-functional is that the stationary condition with respect to μ results in equation (7)

Integrating equation (7) with respect to ν, we obtain the Euler–Lagrange equation

$$\mu_z + \gamma\mu^2\nu + \frac{\partial F}{\partial v} = 0 \tag{10}$$

$$\frac{\delta F}{\delta \nu} = \frac{\partial F}{\partial v} - \frac{\partial}{\partial z}\frac{\partial F}{\partial \mu_z} + \frac{\partial^2}{\partial \tau^2}\frac{\partial F}{\partial \tau\tau} - \cdots \tag{11}$$

Combining equations (10) and (11) and solving for F, we have

$$\frac{\partial F}{\partial v} = -\mu_z - \gamma\mu^2\nu = \frac{1}{2}D\nu_{\tau\tau} + \gamma\nu^3 \tag{12}$$

Integrating both sides of equation (12) with respect to ν, we get

$$F = -\frac{1}{4}D\nu_\tau^2 + \frac{1}{4}\gamma\nu^4 \tag{13}$$

We, therefore, obtain the following needed variation principle

$$J\left(\mu,\nu\right) = \int L_\lambda dzd\tau \tag{14}$$

where the Lagrange multiplier is defined as

$$L_\lambda = \lambda\nu\mu_z - (1-\lambda)\,\nu_z\mu - \frac{1}{4}D\left(\mu_\tau^2 + \nu_\tau^2\right) + \frac{1}{4}\gamma(\mu^2 + \nu^2)^2 + a\mu\mu_z + b\nu\nu_z \qquad (15)$$

Zhang *et al*(2005), where a and b are arbitrary constants whose values can be chosen as $a = -i\lambda$.and $b = i(\lambda - 1)$ in order for equation (15) to be written in the form of Ψ

$$L_\lambda\left(\Psi\right) = i\left(\frac{1}{2} - \lambda\right)\Psi\Psi_z - i\frac{1}{2}\Psi\Psi_z^* - \frac{1}{4}D|\Psi_z|^2 + \frac{1}{4}\gamma|\Psi|^4 \qquad (16)$$

where $\Psi_z^* = (\mu_z - i\nu_z)$.

Selecting the values of $\lambda = 0, \frac{1}{2}$,1 and using the boundary conditions, equation (16) has universality in some sense for nonlinear fiber optics as shown in Fig. 4
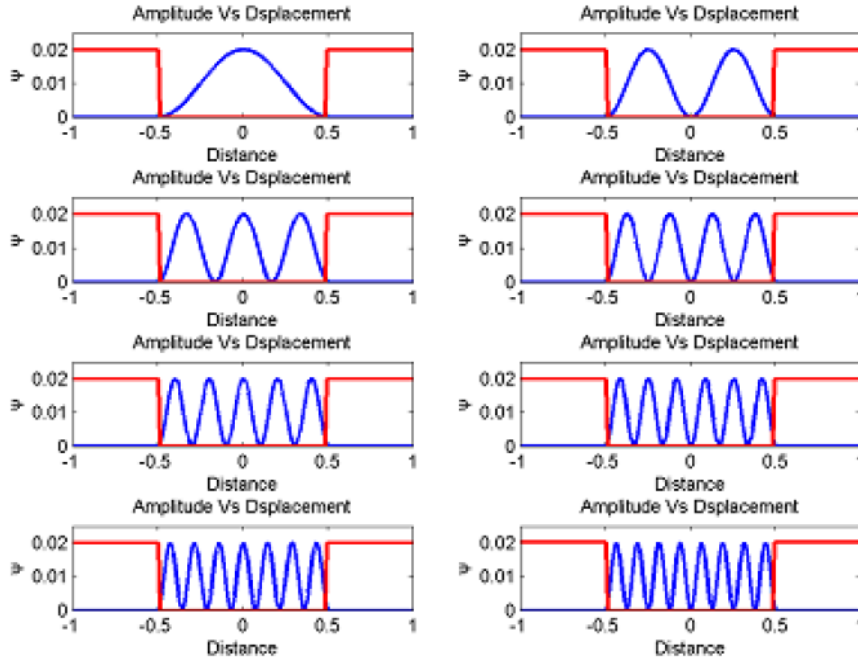
FIGURE 6. Distance Vs Amplitude

3.3. Maxwell Equation

The set of nonlinear differential equations, which is especially suitable for the study of wide-band wavelength-division multiplexed systems of optical communications. The optics of dielectric waveguides is governed by Maxwell's equations, Born (1999).

Due to the material nonlinear effects, a fiber-optic transmission line is a nonlinear channel Haiqing and David (2010).

According to Koshiba (1973), we understand that "a waveguide mode is a transverse field pattern whose amplitude and polarization profile remains constant along the longitudinal direction". The electric and magnetic fields of a mode can be written as

$$E\left(r,t\right) = E_m\left(x,y\right)e^{(i\beta z - i\omega t)} \qquad (17)$$

$$H\left(r,t\right) = H_m\left(x,y\right)e^{(i\beta z - i\omega t)}, \qquad (18)$$

where m is mode index, $E_m(x,y)$ and $H_m(x,y)$ are the mode field patterns and β is the propagation constant of the mode. These equations are differentiated twice to give the TE and TM modes respectively as follows

$$\frac{\partial^2 E_y}{\partial x^2} + \left(k_0^2 - \beta^2\right) E_y = 0 \tag{19}$$

for TE modes,
and

$$\frac{\partial^2 H_y}{\partial x^2} + \left(k_0^2 - \beta^2\right) H_y = \frac{1\partial\varepsilon\partial H_y}{\varepsilon\partial x\partial x} \tag{20}$$

for TM modes where $k_0 = \frac{\omega^2}{c^2} n^2(x)$.

A guided mode can exist only if it satisfies a transverse resonance condition, such that the reflected wave has constructive interference with itself. The transverse component (x-component) of the wave vector inside the core is $h_1 = k_{01}\cos\theta$, where θ is the angle of incidence, and $k_{0i} = \frac{2\pi ni}{\lambda}$, $i = 1, 2, 3$ and the longitudinal component $\beta = k_{01}\cos\theta$. Verma et al (2011)

The transverse for the substrate and cover regions can be defined as $h_2 = k_{02}\cos\theta$, and $h_3 = k_{03}\sin\theta$, where k_{01}, k_{02} and k_{03} are the propagation constants in the respective regions. Applying boundary conditions $\frac{n_1}{n_2}$ and $\frac{n_1}{n_3}$ the interface and using equations (19) and (20) we obtain the eigen value equation for TE and TM mode as follows.

$$tan\left(\frac{h_1 d}{2} - \frac{m\pi}{2}\right) = \frac{\sqrt{V^2 - h_1^2 d^2}}{h_1 d}, \ m = 0, \ 1, \ 2\ldots \tag{21}$$

for TE modes

$$tan\left(\frac{h_1 d}{2} - \frac{m\pi}{2}\right) = \frac{n_1^2}{n_2^2}\frac{\sqrt{V^2 - h_1^2 d^2}}{h_1 d}, \ m = 0, \ 1, \ 2\ldots \tag{22}$$

for TM modes.

FIGURE 7. Maxwell Equation Graph in 3D

From Fig. 4, the left and right hand side of equations (21) and (22) is a function of $h_1 d$ and effective refractive index Neff of a guided mode which will give the value of β and thus effective 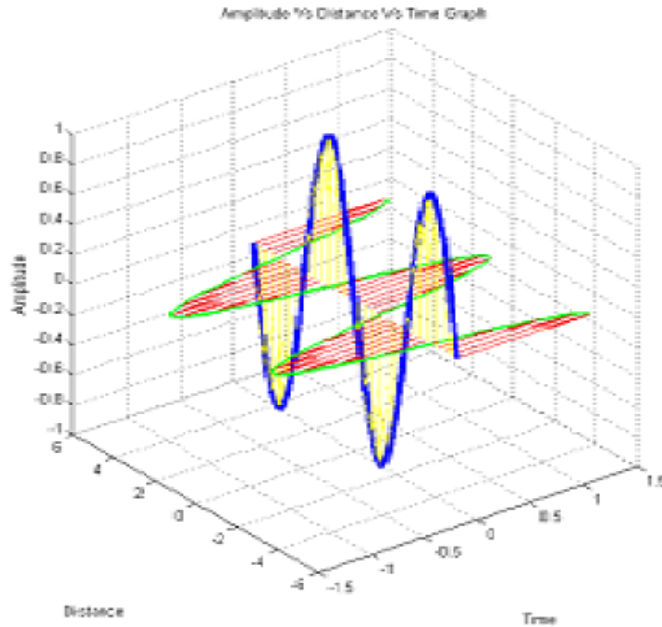index. Applying the boundary conditions; we show the propagation constant for the graph in three dimensions structure, having components for the TE like mode or effective indices are considered in the horizontal structure to get a final structure. The TM field of the vertical structure and the Neff obtained from the analytical calculations.

These results show the changes in shape of these pulses by propagating in optical fiber. From our analysis, we have therefore asserted that sine waves describe many oscillating phenomena. When the wave is damped, each successive peak decreases as time goes on.

Conclusion

This method is applied in ultrahigh-speed all-optical signal processing systems and has the potential to be expanded for more complicated computing functionality. From the graphs, the behaviour of solutions to the PDE's, ODE's and Bessel functions together with their interactions and applications in the fiber optics solutions theory in communication engineering. We model these ODE's and PDE's to describe the impact of dispersion in data transmission. The Bessel functions have a decaying amplitude and therefore recommended for short transmission of data. The Maxwell and Schrodinger equation plots maintain the shapes and therefore recommended for long distance transmission of data.

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Eigenvalue Approach to the Solution of Generalized Thermoelastic
Interactions in an Infinite Body with Cylindrical Cavity

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Abstract

A generalized thermoelastic problem with temperature- dependent modulus of elasticity and thermal conductivity has been considered in an infinite medium with a cylindrical cavity. A fter applying Laplace-transformation the basic equations are presented in the form of a vector-matrix differential equation and then are solved by eigen-value method. Finally, the expressions of radial displacement, temperature and stress distribution are shown graphically for two different cases to compare the situations between the temperature-dependent and temperature-independent material properties in the inverse-Laplace domain.

Eigenvalue Approach to the Solution of Generalized Thermoelastic Interactions in an Infinite Body with Cylindrical Cavity

Abhijit Lahiri and S. Sarkar

Abstract. A generalized thermoelastic problem with temperature- dependent modulus of elasticity and thermal conductivity has been considered in an infinite medium with a cylindrical cavity. A fter applying Laplace-transformation the basic equations are presented in the form of a vector-matrix differential equation and then are solved by eigen-value method. Finally, the expressions of radial displacement, temperature and stress distribution are shown graphically for two different cases to compare the situations between the temperature-dependent and temperature-independent material properties in the inverse-Laplace domain.

Keywords. Generalized Thermoelasticity, Laplace Transform, Vector-matrix differential equation, Eigenvalue Method.

1. Introduction

Lord and Shulman [1] introduced the theory of generalized thermoelasticity with one relaxation time parameter for the special case of an isotropic body. Dhaliwal and Sherief[2] extended this theory to include the anisotropic case. In this theory, a modified law of heat conduction including both the heat flux and its time derivative replaces the conventional Fourier's law. The heat equation associate with this theory is hyperbolic and hence predicts finite speed of propagation for heat waves. The uniqueness of solution for this theory was proved by Ignaczak[3,4], by Dhaliwal and Sherief[2] and by Sherief[5].

Green and Lindsay[6] obtained the theory of thermoelasticity with two relaxation time parameters. In this theory, the classical Fourier's law of heat conduction is not violated when the body under consideration has a centre of symmetry. The uniqueness of solution of this solution was established by Green[7]. The fundamental solution was obtained by Sherief[8].

Indealing with coupled or generalized thermoelastic problems, the solution procedure is to choose a suitable thermoelastic potential function, but this method has certain limitations as discussed by Bahar and Hetnarski[9]. Here we prefer to adopt the eigenvalue method as in Das et.al.[10] and as such, the physical quantities involved in the boundary and initial conditions are directly solvable from the governing equations.

Previously, most of the investigations of thermoelasticity were done under the assumption of

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temperature-independent material properties, but at high temperature the material characteristics such as modulus of elasticity, Poisson's ratio, coefficient of thermal expansion and the thermal conductivity are no longer constants[11]. In recent years, it has become necessary to take into account the actual behavior of material characteristics.

In this paper, we have considered a infinite medium with a cylindrical cavity where the modulus of elasticity and thermal conductivity are temperature dependent and comparisons are made graphically between the temperature dependent and temperature independent material properties.

NOMENCLATURE

$\lambda, \mu =$ Lamè constants.
$u =$ Displacement component.
$t =$ Time variable.
$\sigma_{ij} =$ Stress component.
$T =$ Absolute temperature.
$T_0 =$ Reference temperature.
$\rho =$ Mass density.
$C_e =$ Specific heat.
$K =$ Coefficient of thermal conductivity.
$\kappa =$ Coefficient of thermal diffusivity.
$\gamma = (3\lambda + 2\mu)\alpha_t$.
$\alpha^* =$ Empirical material constant.
$\tau_0 =$ Thermal relaxation time parameter.
$H(t) =$ Heaviside unit step function.

2. Formulation of the problem

We have taken into account the generalized thermoelasticity with one relaxation time in an isotropic infinite medium which has a cylindrical cavity of radius R.
We use cylindrical co-ordinate system (r, ψ, z) with z-axis lying along the axis of the cylinder.
Due to symmetry, all functions are dependent only on r and t.
i.e. if $\vec{u} = (u_r, u_\psi, u_z)$ be the displacement vector, then
$u_r = u(r, t)$; $u_\psi(r, t) = 0 = u_z(r, t)$
In this paper, the modulus of elasticity and the heat conductivity are taken to be temperature dependent as
$\lambda = \lambda_0 f(T); \mu = \mu_0 f(T); K = K_0 f(T); \gamma = \gamma_0 f(T);$
where $f(T) \approx f(T_0) = 1 - \alpha^* T_0 = \frac{1}{\alpha_T}$ considering $|\frac{T-T_0}{T_0}| << 1$
The equation of motion and the heat conduction equations are :

$$\rho \alpha_T \ddot{u} = (\alpha_0 + 2\mu_0)\frac{\partial e}{\partial r} - \gamma_0 \frac{\partial \theta}{\partial r} \tag{1}$$

and

$$\nabla^2 \theta = (\frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2})(\frac{\theta}{\kappa} + \frac{\gamma_0 T_0 e}{K_0}) \tag{2}$$

where $e = \frac{1}{r}\frac{\partial(ru)}{\partial r}$; $\theta = T - T_0$; $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}$

The stress components are :

$$\alpha_T \sigma_{rr} = 2\mu_0 \frac{\partial u}{\partial r} + \lambda_0 e - \gamma_0 \theta$$

$$\alpha_T \sigma_{\psi\psi} = 2\mu_0 \frac{u}{r} + \lambda_0 e - \gamma_0 \theta$$

$$\alpha_T \sigma_{zz} = \lambda_0 e - \gamma_0 \theta$$

$$\sigma_{zr} = \sigma_{\psi r} = \sigma_{z\psi} = 0 \tag{3}$$

We define the following non-dimensional variables :

$r^{'} = \sqrt{\frac{\lambda_0 + 2\mu_0}{\rho}}\frac{r}{\kappa}$; $u^{'} = \sqrt{\frac{\lambda_0 + 2\mu_0}{\rho}}\frac{u}{\kappa}$; $t^{'} = \sqrt{\frac{\lambda_0 + 2\mu_0}{\rho}}\frac{t}{\kappa}$; $\tau_0^{'} = \sqrt{\frac{\lambda_0 + 2\mu_0}{\rho}}\frac{\tau_0}{\kappa}$; $R^{'} = \sqrt{\frac{\lambda_0 + 2\mu_0}{\rho}}\frac{R}{\kappa}$;

$\theta^{'} = \frac{\theta}{T_0}$; $\sigma^{'} = \frac{\sigma}{\mu_0}$

Using the above non-dimensional quantities in equations (1), (2) and (3), we get(omiting the primes) :

$$\alpha_T \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} - a\frac{\partial \theta}{\partial r} \tag{4}$$

$$\frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r}\frac{\partial \theta}{\partial r} - (\dot{\theta} + \tau_0 \ddot{\theta}) = g(\frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2})(\frac{u}{r} + \frac{\partial u}{\partial r}) \tag{5}$$

$$\alpha_T \sigma_{rr} = \beta^2 \frac{\partial u}{\partial r} + (\beta^2 - 2)\frac{u}{r} - b\theta$$

$$\alpha_T \sigma_{\psi\psi} = (\beta^2 - 2)\frac{\partial u}{\partial r} + \beta^2 \frac{u}{r} - b\theta$$

$$\alpha_T \sigma_{zz} = (\beta^2 - 2)(\frac{\partial u}{\partial r} + \frac{u}{r} - b\theta \tag{6}$$

3. Method of Solution

Consider the definition of Laplace transform :

$$\bar{T}(r, p) = \int_0^\infty T(r, t)exp(-pt)dt \tag{7}$$

Using the above Laplace transform on time t to the equations (4), (5) and (6), we get:

$$\alpha_T p^2 \bar{u} = \frac{d^2 \bar{u}}{dr^2} + \frac{1}{r}\frac{d\bar{u}}{dr} - \frac{\bar{u}}{r^2} - a\frac{d\bar{\theta}}{dr} \tag{8}$$

$$\frac{d^2 \bar{\theta}}{dr^2} + \frac{1}{r}\frac{d\bar{\theta}}{dr} - (p + \tau_0 p^2)\bar{\theta} = g(p + \tau_0 p^2)(\frac{\bar{u}}{r} + \frac{d\bar{u}}{dr}) \tag{9}$$

$$\alpha_T \bar{\sigma}_{rr} = \beta^2 \frac{d\bar{u}}{dr} + (\beta^2 - 2)\frac{\bar{u}}{r} - b\bar{\theta}$$

$$\alpha_T \bar{\sigma}_{\psi\psi} = (\beta^2 - 2)\frac{d\bar{u}}{r} + \beta^2 \frac{\bar{u}}{r} - b\bar{\theta}$$

$$\alpha_T \bar{\sigma}_{zz} = (\beta^2 - 2)(\frac{d\bar{u}}{dr} + \frac{\bar{u}}{r} - b\bar{\theta} \tag{10}$$

Differentiating (9) with respect to r and using (8) in the resulting equation :

$$\frac{d^3 \bar{\theta}}{dr^3} + \frac{1}{r}\frac{d^2 \bar{\theta}}{dr^2} - \frac{1}{r^2}\frac{d\bar{\theta}}{dr} = p(1 + \tau_0 p)(1 + ag)\frac{d\bar{\theta}}{dr} + \alpha_T g p^3 (1 + \tau_0 p)\bar{u} \tag{11}$$

Now we write equations (8) and (11) in the form of a vector-matrix differential equation as :

$$L\underline{V} = A\ \underline{V} \tag{12}$$

where $L \equiv \frac{d^2}{dR^2} + \frac{1}{R}\frac{d}{dR} - \frac{1}{R^2}$ is a Bessel operator.

$$\underline{v} = \left[\overline{u}\ \frac{d\overline{\theta}}{dr}\right]^T \ ;\ A = \left[\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array}\right]$$

$a_{11} = \alpha_T p^2\ ;\ a_{12} = a\ ;\ a_{21} = \alpha_T g p^3 (1 + \tau_0 p)\ ;\ a_{22} = p(1 + \tau_0 p)(1 + \varepsilon)$

Let $\lambda_i = \alpha_i^2$, $i = 1, 2$ be the two eigen values of the matrix A which can be determined from the following characteristic equation :-

$\alpha^4 - (a_{11} + a_{22})\alpha^2 + (a_{11}a_{22} - a_{12}a_{21}) = 0$

The eigen vectors corresponding to the eigen values $\lambda_i = \alpha_i^2$, $i = 1, 2$ are respectively :-

$V_i = \left[\begin{array}{c} -a_{12} \\ a_{11} - \alpha_i^2 \end{array}\right]$

Hence by Eigenvalue Method [Appendix], the solution of $\underline{v}$ can be written as :-

$$\underline{v} = \left[\begin{array}{c} \overline{u} \\ \frac{d\overline{\theta}}{dr} \end{array}\right] = \sum_{i=1}^{2}[A_i V_i K_1(\alpha_i r) + B_i V_i I_1(\alpha_i r)] \tag{13}$$

where K_1 and I_1 are the modified Bessel Functions of second kind of order 1.

The radial displacement and the temperature in the Laplace transformed domain which are both bounded at infinity, are now can be written as :-

$$\overline{u} = -a_{12}[A_1 K_1(\alpha_1 r) + A_2 K_1(\alpha_2 r)] \tag{14}$$

$$\overline{\theta} = -[A_1 \frac{a_{11} - \alpha_1^2}{\alpha_1} K_0(\alpha_1 r) + A_2 \frac{a_{11} - \alpha_2^2}{\alpha_2} K_0(\alpha_2 r)] \tag{15}$$

4. Boundary Conditions

To calculate the unknown constants A_1 and A_2 , we use the following boundary conditions on the internal boundary $r = R$:-

Case I:-
In this case the cavity surface is assumed to be maintained at zero temperature and is subjected to a ramp-type boundary load , i.e.

$$\sigma_{rr}(R, t) = -\sigma_0 H(t)\ ,\ \theta(R, t) = 0 \tag{16}$$

Using Laplace Transformation on above :-

$$\overline{\sigma}_{rr}(R, p) = -\frac{\sigma_0}{p}\ ,\ \overline{\theta}(R, p) = 0 \tag{17}$$

Case II:-
Considering the thermoelastic interactions when the surface of the cavity is stress-free and kept at a temperature $\theta(R, t)$,then the boundary condition takes the form

$$\sigma_{rr}(R, t) = 0\ ,\ \theta(R, t) = \theta_0 e^{-\omega t} \tag{18}$$

Using Laplace Transformation on above :-

$$\overline{\sigma}_{rr}(R, p) = 0\ ,\ \overline{\theta}(R, p) = \frac{\theta_0}{p + \omega} \tag{19}$$

5. Numerical Results

For our final result we have to find out the Laplace-inversion of radial displacement , temperature and stress distribution which are very complicated in nature.

To evaluate these we have used the Zakian method [12] . For our numerical calculation we have chosen the Copper material . The values of the constants are given by :-

$\lambda_0 = 7.76$; $\mu_0 = 3.86$; $\gamma_0 = (3\lambda_0 + 2\mu_0)\alpha_t = 55.18$; $\alpha_t = 1.78$; $K_0 = 386$; $c_E = 3.831$; $\rho = 8954$; $T_0 = 293K$; $\beta^2 = 4.01$; $b = 0.042$; $a = 0.0105$; $g = 1.61$; $\tau_0 = 0.01$; $\varepsilon = ag = 0.017$; $R = 1$; $t = 0.3$; $\sigma_0 = 1.2$; $\theta_0 = 1.5$; $\omega = 0.5$

Finally the expressions of radial displacement u , temperature θ and stress distribution σ_{rr} for the above two cases are presented graphically (shown below) where the curves are plotted for different values of α_T (viz. 0.3 , 1 , 1.7). From the following graphs we observe that :-

i) we see that the three quantities stated above have their maximum values (absolute value) for temperature independent case i.e. when $\alpha_T = 1$ for both case I and case II.

ii) for case I :-
• the distribution of displacement has greater absolute values for $\alpha_T = 1.7$ (i.e. $\alpha_T > 1$) than for $\alpha_T = 0.3$(i.e. $\alpha_T < 1$).
• the temperature distribution has values near about zero for $\alpha_T = 0.3$.
• for stress distribution the absolute values are more or less same for $\alpha_T > 1$ and $\alpha_T < 1$.

iii) for case II :-
• the displacement distribution has greater absolute values for $\alpha_T < 1$ than for $\alpha_T > 1$ which is the reverse of case I.
• for temperature distribution the absolute values are almost same for both $\alpha_T > 1$ and $\alpha_T < 1$.
• the radial tress distribution has greater absolute values for $\alpha_T < 1$ than for $\alpha_T > 1$.

6. Tables, figures and list

Case I : Distribution of Displacement

Case I : Distribution of Temperature

Case I : Distribution of Radial Stress

Case II : Distribution of Displacement

Case II : Distribution of Temperature

Case II : Distribution of Radial Stress
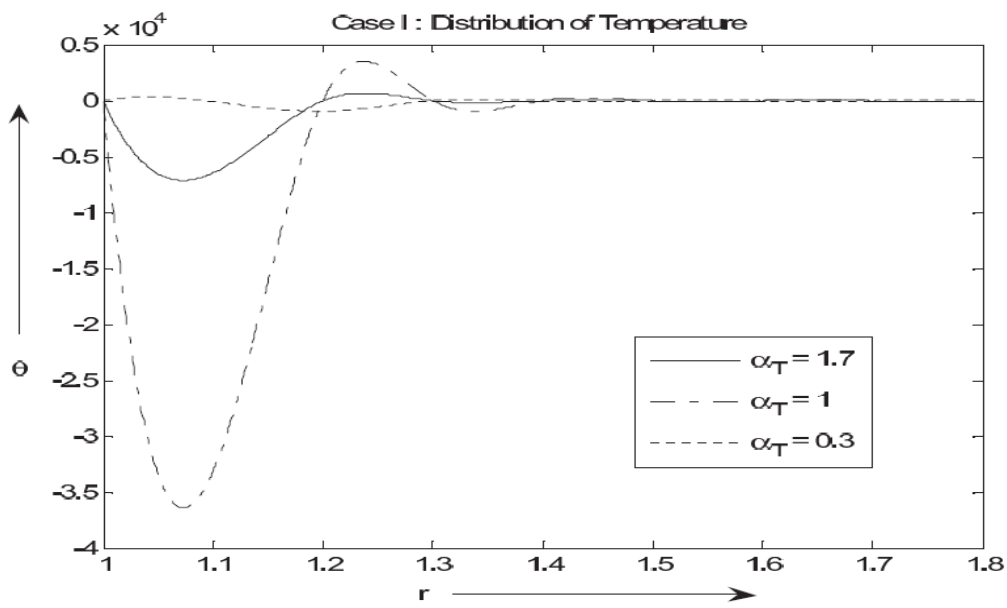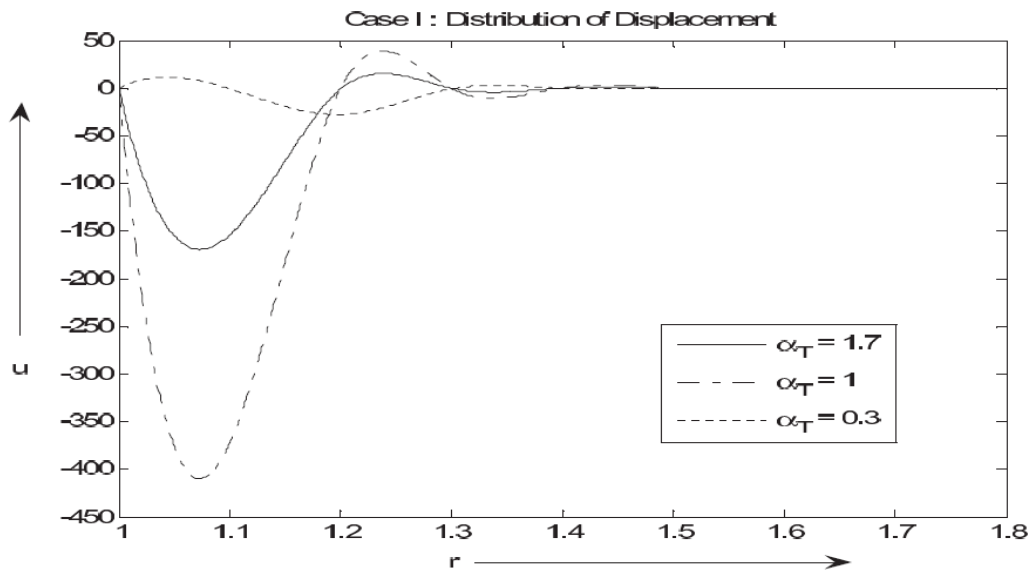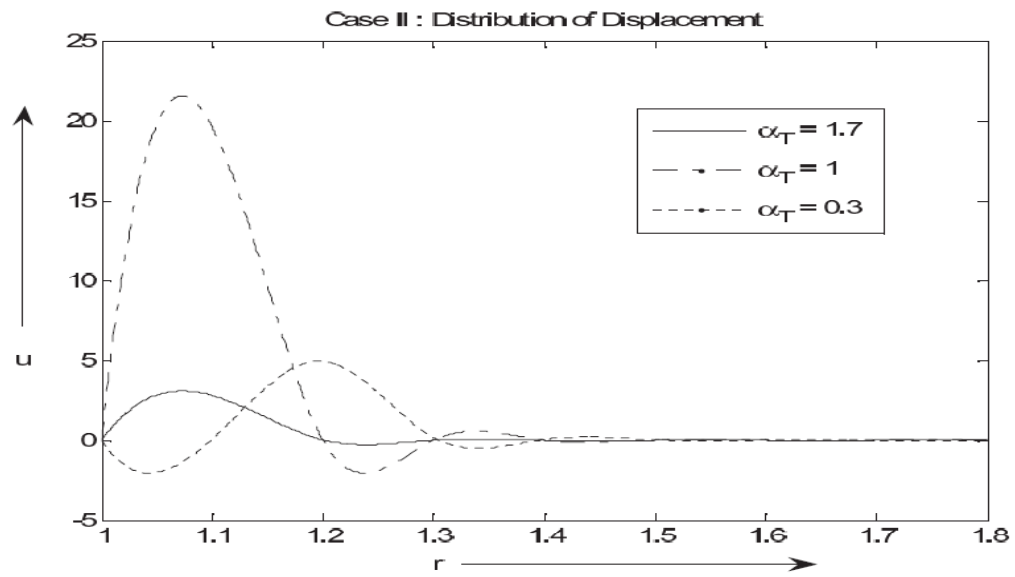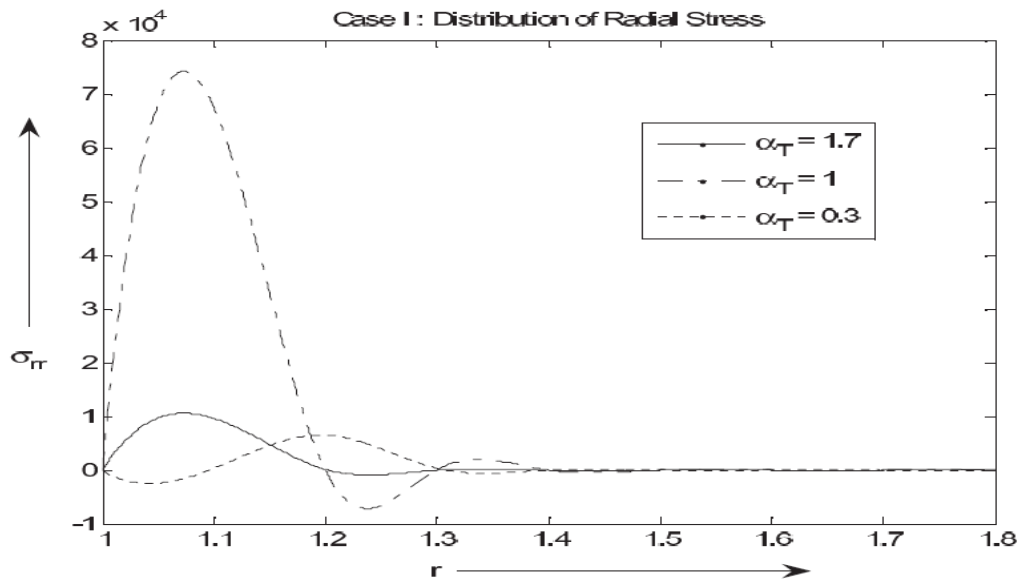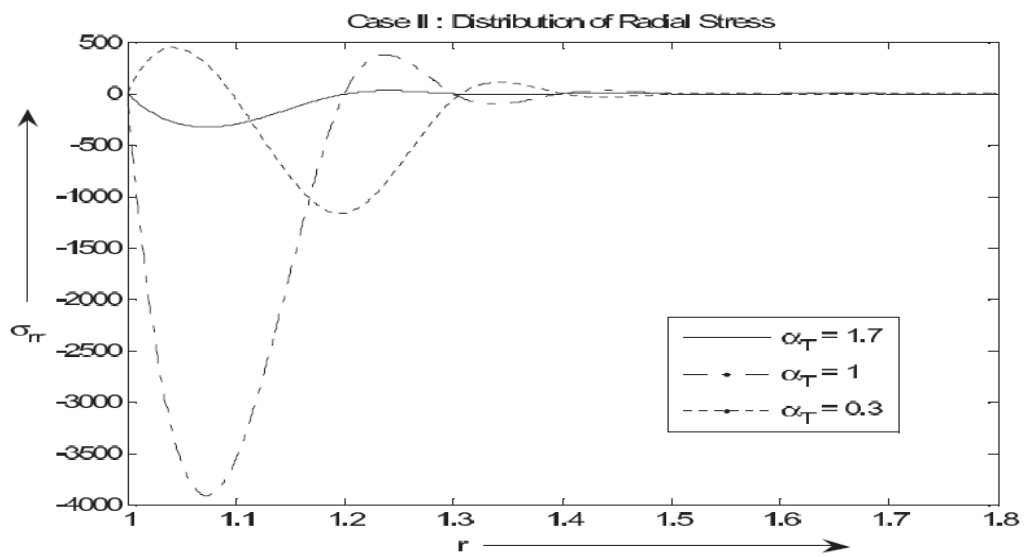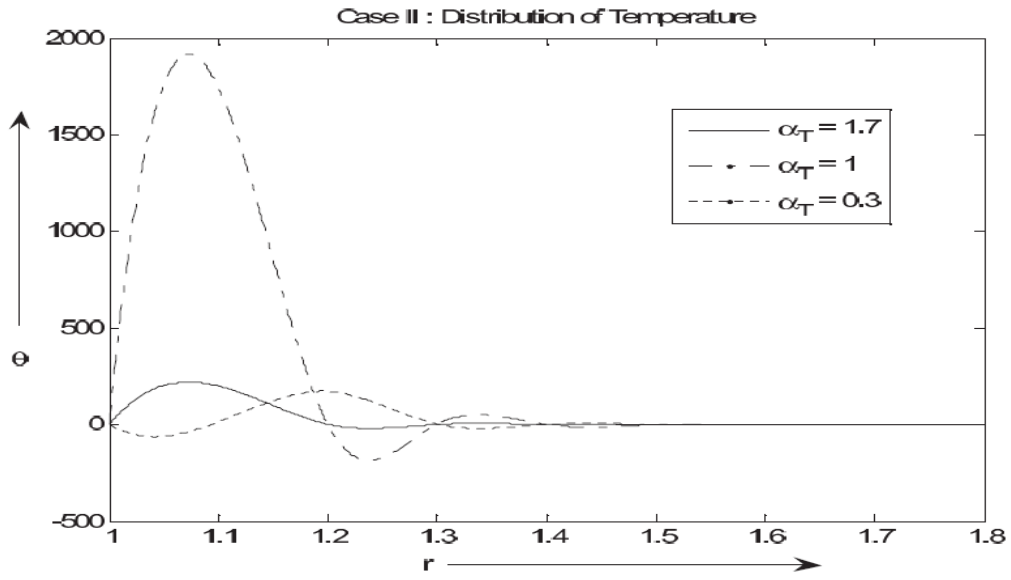
Appendix

Consider the differential equation in the form :

$$L\underline{V} = A\ \underline{V} \tag{20}$$

where $L \equiv \frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} - \frac{n^2}{x^2}$ is a Bessel operator.
Let

$$A = V\Lambda V^{-1} \tag{21}$$

where $\Lambda = \begin{bmatrix} \lambda_1 & ... & 0 \\ ... & \lambda_2 & ... \\ ... & ... & ... \\ 0 & ... & \lambda_n \end{bmatrix}$

is a diagonal matrix whose elements $\lambda_1.\ \lambda_2.\ ....,\lambda_n$ are the distinct eigenvalues of A. Let $\underline{V}_1$. $\underline{V}_2.\ .....,\underline{V}_n$ be the eigenvectors of A corresponding to $\lambda_1.\ \lambda_2.\ ....,\lambda_n$ respectively, and

$$V = [\underline{V}_1\ \underline{V}_2\ ...\ \underline{V}_n] \tag{22}$$

Substituting (21) in (20) and premultiplying by V^{-1}, we get

$$L\underline{y} = \Lambda\underline{y}\ ;\ \underline{y} = V^{-1}\underline{v} \tag{23}$$

as a system of decoupled equations.
A typical rth equation of (23) is

$$Ly_r = \lambda_r y_r$$

$$\frac{d^2 y_r}{dx^2} + \frac{1}{x}\frac{dy_r}{dx} - (\lambda_r + \frac{n^2}{x^2})y_r = 0 \tag{24}$$

Case (i)

When $\lambda_r = \alpha_r^2$, the solution of equation (24) can be written as,

$$y_r = A_r K_n(\alpha_r x) + B_r I_n(\alpha_r x) \tag{25}$$

n is integer and A_r, B_r are constants. K_n, I_n are modified Bessel functions of the second kind of order n.

Case (ii)

When $\lambda_r = -\alpha_r^2$, the solution can be written as

$$y_r = A_r J_n(\alpha_r x) + B_r Y_n(\alpha_r x) \tag{26}$$

n is integer and J_n, Y_n are Bessel functions of the first kind of order n.
Hence the complete solution in this case can be written as $\underline{v} = \sum_{r=1}^{n} V_r Y_r$.

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Cocycle of multifunctions

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Abstract

In this paper we introduce cocycles of multifunctions, and we study the concept of attractors for them by using of semibornologies. We define a kind of conjugate relation on them, and we show that a conjugacy takes an attractor to an attractor. We consider the role of semibornology on the existence of new attractors.

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MohammadReza Molaei

Abstract. In this paper we introduce cocycles of multifunctions, and we study the concept of attractors for them by using of semibornologies. We define a kind of conjugate relation on them, and we show that a conjugacy takes an attractor to an attractor. We consider the role of semibornology on the existence of new attractors.

Mathematics Subject Classification (2010). 37H05 (primary), 37A20 (secondary).

Keywords. Cocycle, Semibornology, Non-autonomous dynamical system, Iterated function system.

1. Introduction

Cocycles in continuous or discrete cases play an important role in the modeling of nature phenomena [2, 5]. Here we extend this notion to multifunctions. We introduce the concept of attractors for cocycles of multifunctions. If in a cocycle all the multifunctions are equal, then this notion is equivalent to the concept of attractors for multifunctions, which has been studied first by Lasota and Myjak [6]. We present an equivalence relation on cocycles, and we show that if two cocycles are conjugate with a conjugacy h, then h takes an attractor of the first cocycle to an attractor of the second one. We consider the effect of semibornology on an attractor of a cocycle. We present a class of cocycles by using of top spaces.

2. Non-autonomous dynamical systems created by multifunctions

Suppose X_1 and X_2 are two sets, a relation $F_1 : X_1 \to X_2$ is called a multifunction if X_1 is the domain of F_1. If $X = \{X_i\}_{i=0}^{\infty}$ is a sequence of sets and $F = \{F_i \ : \ F_i : X_i \to X_{i+1}\}_{i=0}^{\infty}$ is a sequence of multifunctions, then we say that (X, F) is a cocycle of multifunctions. In this kind of dynamics we have many choices to continue a process. This situation occurs in the meanings of sentences. For example saying one sentence by a person can have different meanings for the others, and the new sentences of each of them, also can have different meanings for the others. To continue this process we find a cocycle of multifunctions.

When each F_i is a function, then (X, F) is a non-autonomous dynamical system [4]. So generally a cocycle of multifunctions creates a class of non-autonomous dynamical systems.

Iterated function systems [1] are also a special case of cocycle of multifunction. In a cocycle if $X_0 = X_1 = X_2 = ...$, $F_0 = F_1 = F_2 = ...$, and for each $x \in X_0$, the cardinality of $F_0(x)$ is a constant natural number, then it is an iterated function system.

We assume that β is a semibornology for $\bigcup_{i=0}^{\infty} X_i$ [9], this means that β is a cover for $\bigcup_{i=0}^{\infty} X_i$, and it is closed under finite unions. A semibornology β is called a bornology [3] if any subset of each member of β is a member of it. For example the set of bounded subsets of a metric space is a bornology for it, and the set of closures of the subsets of a topological space is a semibornology for it.

We assume that $(A_i)_{i=0}^{\infty}$ is a sequence with $A_i \subseteq X_i$. The lower bound of $(A_i)_{i=0}^{\infty}$ is denoted by LiA_i and it is the set of $x \in \bigcup_{i=0}^{\infty} X_i$ such that for each $B \in \beta$ with $x \in B$ there is $i_0 \in N \cup \{0\}$ such that $B \cap A_i \neq \emptyset$ for all $i \geq i_0$. The upper bound of $(A_i)_{i=0}^{\infty}$ is denoted by LsA_i, and it is the set of $x \in \bigcup_{i=0}^{\infty} X_i$ such that for all $B \in \beta$ with $x \in B$ there exists $i_0 \in N \cup \{0\}$ so that $B \cap A_i \neq \emptyset$ for infinitely many $i \geq i_0$. We say that $(A_i)_{i=0}^{\infty}$ has bounded limit if $LsA_i = LiA_i$. The bounded limit of (A_i) in the case of existence is denoted by LbA_i, and it is defined by $LbA_i = LsA_i$.

Suppose $h : \bigcup_{i=0}^{\infty} X_i \to \bigcup_{i=0}^{\infty} X_i$ is a bijection with $h_i(X_i) = X_i$, where h_i is the restriction of h to X_i, and $i \in \{0, 1, 2, ...\}$. We say that h is a semibornological bijection if $h(B) \in \beta$, and $h^{-1}(B) \in \beta$, for all $B \in \beta$.

Theorem 2.1. *If h is a semibornological bijection, then $h(LiA_i) = Lih_i(A_i)$, $h(LsA_i) = Lsh_i(A_i)$, and $h(LbA_i) = Lbh_i(A_i)$.*

  Proof. We only prove that $h(LiA_i) = Lih_i(A_i)$. The other parts can prove by similar method.

$$x \in LiA_i \Leftrightarrow \text{ For all } h(B) \in \beta \text{ with } x \in h(B) \text{ there is } i_0 \text{ such that}$$
$$h(B) \cap h_i(A_i) \neq \emptyset \text{ for all } i \geq i_0. \Leftrightarrow \text{For all } B \in \beta \text{ with } h^{-1}(x) \in B \text{ there is } i_0$$
$$\text{such that } h(B \cap A_i) \neq \emptyset \text{ for all } i \geq i_0. \Leftrightarrow \text{For all } B \in \beta \text{ with } h^{-1}(x) \in B \text{ there is } i_0$$
$$\text{such that } B \cap A_i \neq \emptyset \text{ for all } i \geq i_0. \Leftrightarrow h^{-1}(x) \in LiA_i.$$

Thus $h(LiA_i) = Lih_i(A_i)$.

  ∎

3. Attractors

We assume that X and F are the sets of previous section and β is a semibornology for $\bigcup_{i=0}^{\infty} X_i$.

Definition 3.1. *A non-empty subset A of $\bigcup_{i=0}^{\infty} X_i$ is called an attractor for a cocycle (X, F) if for all $\emptyset \neq B \in \beta$ with $A \subseteq B$, we have $A = LbF_i(B \cap X_i)$.*

Example 3.2. *For $i \in \{0, 1, 2, ...\}$ we take $X_i = R^{i+1}$, and we define*

$$F_i : R^{i+1} \to R^{i+2}$$

$$(x_1, x_2, x_3, ..., x_{i+1}) \mapsto \{(\frac{1}{2^j} x_1, \frac{1}{2^j} x_2, ..., \frac{1}{2^j} x_{i+1}, \frac{1}{2^j} x_{i+1}) \ : \ j \geq i\}.$$

If $\beta = \{\bigcup_{i=0}^{\infty} B_i \ : \ B_i$ is the closure of an open set in $R^{i+1}\}$, then $\{0, (0, 0), (0, 0, 0), ...\}$ is the attractor of (X, F).

Now we assume that (X, F) and (X, G) are two cocycles of multifunctions and β is a semibornology for $\bigcup_{i=0}^{\infty} X_i$. Moreover we assume that $h : \bigcup_{i=0}^{\infty} X_i \to \bigcup_{i=0}^{\infty} X_i$ is a bijection such that $h_i(X_i) = X_i$, where h_i is the restriction of h to X_i, and $i \in \{0, 1, 2, ...\}$. We say that (X, F) is conjugate to (X, G) under the conjugacy h if h is a semibornological bijection, and for given $i \in \{0\} \cup N$, the following diagram commutes.

$$
\begin{array}{ccc}
X_i & \xrightarrow{F_i} & X_{i+1} \\
h_i \downarrow & & \downarrow h_{i+1} \\
X_i & \xrightarrow{G_i} & X_{i+1}
\end{array} \; .
$$

Theorem 3.3. *If A is an attractor for (X, F), and if (X, F) is conjugate to (X, G) under a conjugacy h then $h(A)$ is an attractor for (X, G).*

       Proof. Suppose $B \in \beta$ with $h(A) \subseteq B$ be given. Since $h^{-1}(B) \in \beta$, and $A \subseteq h^{-1}(B)$, then $A = LbF_i(h^{-1}(B) \cap X_i)$. Theorem 2.1. implies $h(A) = Lbh_{i+1}(F_i(h^{-1}(B) \cap X_i))$. Thus

$$h(A) = LbG_i(h_i(h^{-1}(B) \cap X_i)) = LbG_i(B \cap X_i).$$

So $h(A)$ is an attractor for (X, G).

                                                              ■

If (X, F) is a cocycle of multifunctions and β is a semibornology for $X = \bigcup_{i=0}^{\infty} X_i$, then we define $C(\beta)$ by

$$C(\beta) = \{D \in \beta \; : \; For \; given \; y \in D \; and \; for \; all \; x \; if \; there \; is \; j \; such \; that \; y \in F_j(x)$$

$$then \; there \; is \; B \in \beta \; with \; x \in B \; such \; that \; F_i(B \cap X_i) = D \cap X_{i+1} \; for \; all \; i\}.$$

If we denote an attractor of (X, F) with respect to a semibornology β by A^β, then we have the next theorem.

Theorem 3.4. *If $C(\beta)$ is a cover for X, then $C(\beta)$ is a semibornology for X. Moreover if A^β is an attractor and if there is $E \in C(\beta)$ such that $A^\beta \subseteq E$ then there is an attractor $A^{C(\beta)}$ such that $A^\beta \subseteq A^{C(\beta)}$.*

       Proof. Let $D_1, D_2 \in C(\beta)$ and $y \in D_1 \cup D_2$ be given. Without loss of generality we assume that $y \in D_1$. We choose $z \in D_2$, then there exist $a \in X$ and $l \in N \cup \{0\}$ such that $z \in F_l(a)$. If $y \in F_j(x)$ for some j, then there exist $B_1, B_2 \in \beta$ such that $x \in B_1$ and $a \in B_2$ and $F_i(B_j \cap X_i) = D_j \cap X_{i+1}$ for all $i \in N \cup \{0\}$. For given $i \in N \cup \{0\}$, $F_i((B_1 \cup B_2) \cap X_i) = F_i(B_1 \cap X_i) \cup F_i(B_2 \cap X_i) = (D_1 \cap X_{i+1}) \cup (D_2 \cap X_{i+1}) = (D_1 \cup D_2) \cap X_{i+1}$. Since $x \in B_1 \cup B_2 \in \beta$, then $D_1 \cup D_2 \in C(\beta)$. So it is a semibornology for X. If $x \in A^\beta$, then for all $B, E \in \beta$ with $x \in B$, and $A^\beta \subseteq E$ there is i_0 such that for infinitely $i \geq i_0$, $B \cap F_i(X \cap E) \neq \emptyset$, and there is $k \in N$ so that for all $i \geq i_0 + k$, $B \cap F_i(X \cap E) \neq \emptyset$. Since $C(\beta) \subseteq \beta$, then for all $B, E \in C(\beta)$ with $x \in B$, and $A^\beta \subseteq E$ there is i_0 such that for infinitely $i \geq i_0$, $B \cap F_i(X \cap E) \neq \emptyset$, and there is $k \in N$ so that for all $i \geq i_0 + k$, $B \cap F_i(X \cap E) \neq \emptyset$. Thus $x \in A^{C(\beta)}$.

                                                              ■

Theorem 3.5. *Suppose $x \in A^\beta \cap X_j$ for some j. If $C(\beta)$ is a cover for X, then $F_j(x) \subseteq A^{C(\beta)}$.*

Proof. Let $y \in F_j(x)$, $E \in C(\beta)$ with $y \in E$, and $D \in C(\beta)$ with $A^{C(\beta)} \subseteq D$ be given. Then there is $B \in \beta$ with $x \in B$ such that $F_i(B \cap X_i) = E \cap X_{i+1}$ for all $i \in N \cup \{0\}$. Since $D \in C(\beta)$, then there is $S \in \beta$ such that $F_i(S \cap X_i) = D \cap X_{i+1}$ for all $i \in N \cup \{0\}$. We know that $x \in A^\beta$

so there is i_0 such that for infinitely $i \geq i_0$, $B \cap F_{i-1}(S \cap X_{i-1}) \neq \emptyset$, and there is $k \in N$ so that for all $i \geq i_0 + k$, $B \cap F_{i-1}(S \cap X_{i-1}) \neq \emptyset$. We have

$$E \cap F_i(D \cap X_i) = E \cap X_{i+1} \cap F_i(D \cap X_i) = F_i(B \cap X_i) \cap F_i(F_{i-1}(S \cap X_{i-1}))$$

$$\supseteq F_i(B \cap X_i \cap F_{i-1}(S \cap X_{i-1})).$$

Thus for infinitely $i \geq i_0$, $E \cap F_i(D \cap X_i) \neq \emptyset$, and for all $i \geq i_0 + k$, $E \cap F_i(D \cap X_i) \neq \emptyset$. Hence $y \in A^{C(\beta)}$.

$\blacksquare$

4. Cocycle of multifunctions created by a family of diffeomorphisms on top spaces

We begin this section by recalling the definition of a top space [7]. A smooth manifold T is called a top space if it has a binary smooth operator

$$\begin{aligned} m : T \times T &\longrightarrow T \\ (a,b) &\mapsto ab \end{aligned}$$

with the following conditions.
(i) (T, m) is a semigroup;
(ii) For all $a \in T$, there is a unique $e(a) \in T$ such that $ae(a) = e(a)a = a$;
(iii) For all $a \in T$ there is $a^{-1} \in T$ such that $aa^{-1} = a^{-1}a = e(a)$, and the mapping

$$\begin{aligned} in : T &\longrightarrow T \\ a &\mapsto a^{-1} \end{aligned}$$

is a smooth mapping.

Clearly each Lie group is a top space, but the converse may not be true. We can use of Rees matrix semigroups [10] to construct top spaces which are not Lie groups. In fact if G is a Lie group, M and N are two smooth manifolds, and $p : M \times N \to G$ is a smooth mapping then $N \times G \times M$ with the product $(n, a, s)(k, b, l) = (n, ap(s, k)b, l)$ is a top space [8].

Now we assume that $T = \{T_i\}_{i=0}^{\infty}$ is a family of top spaces. We take

$$f = \{f_i \mid f_i : T_i \to T_i \text{ is a } diffeomorphism\}_{i=0}^{\infty}, \text{ and}$$

$$g = \{g_i \mid g_i : T_i \to T_{i+1} \text{ is a smooth map}\}_{i=0}^{\infty}.$$

For given i we define a multifunction $F_i : T_i \to T_{i+1}$ by $F_i(t) = \{g_{i,j}(t) \mid j \in Z\}$ where

$$g_{i,j}(t) = \begin{cases} g_i(f_i^{j-1}(t))...g_i(f_i(t))g_i(t) & \text{if } j > 0 \\ g_i(f_i^{-j}(t))...g_i(f_i^{-1}(t)) & \text{if } j < 0 \\ e_{i+1}(g_i(t)) & \text{if } j = 0 \end{cases} .$$

If $F = \{F_i\}_{i=0}^{\infty}$, then (T, F) is a cocycle of multifunctions.

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Boundary element method for solving high frequency scattering problems for obstacles with no corners

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Abstract

We consider scattering of a time-harmonic acoustic incident plane wave by a sound soft smooth object with Lipschitz boundary. The application of conventional boundary or finite element methods, have computational cost that grows linearly respect to the frequency of the incident wave. Recent research has been devoted in finding methods which does not loose robustness as frequency of the incident wave increases. *Arden, Chandler-Wilde and Langdon* proposed a collocation method to solve a high frequency scattering by convex polygons. They use a boundary element method, and incorporating products of plane wave basis functions with piecewise polynomials supported on a graded mesh into approximation space. They demonstrated via numerical experiments the number of degrees of freedom required to achieve a prescribed level of accuracy grows only logarithmically with respect to frequency. Here we proposed a collocation method for high frequency scattering by smooth objects (objects with no corners, e.g. a circle). We applied same approximations as theirs, but employing uniform mesh. We demonstrate through numerical experiments the logarithmical grow of the solutions as frequency increases, with much reduced computational cost.

Boundary element method for solving high frequency scattering problems for obstacles with no corners

M. Mokgolele

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Mathematics Subject Classification (2010). 47A40, 65L60, 42B20 .

Keywords. Scattering problems, Collocation method, Osillatory integrals.

1. Introduction

We consider the two-dimensional problem of scattering of a time-harmonic acoustic incident plane wave

$$u^i(\mathbf{x}) = e^{ik\mathbf{x}\cdot\mathbf{d}}, \text{ in } D := \mathbb{R}^2 \backslash \bar{\Omega},$$

by a smooth convex sound-soft obstacle Ω, with Lipschitz boundary Γ. Here $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, $\mathbf{d} = (\sin\theta, -\cos\theta) \in \mathbb{R}^2$ is a unit vector representing the direction of the incident field, and the frequency of the incident wave is proportional to the wavenumber $k > 0$. The scattered field $u^s := u - u^i \in C^2(\bar{D})$ (where u and u^i denote the total and incident field respectively) satisfies the *Helmholtz equation*

$$\Delta u^s + k^2 u^s = 0, \text{ in } D, \tag{1}$$

We consider a sound-soft boundary condition,

$$u = 0, \text{ or } u^s = -u^i, \text{ on } \Gamma \tag{2}$$

we also need the Sommerfeld radiation condition,

$$\lim_{r \to \infty} r^{\frac{1}{2}} \left(\frac{\partial u^s}{\partial r} - \mathrm{i}ku^s \right) = 0, \tag{3}$$

where $r := |\mathbf{x}|$ and the limit holds uniformly in $\mathbf{x}/|\mathbf{x}|$. The Sommerfeld radiation condition is essential to scattering problems because it ensures that the scattered field is not reflected back from infinity.

We can reformulate (1 - 3) using Green's theorems [12] and following the usual coupling procedure to obtain a Fredholm integral equation of the second kind for $\frac{\partial u}{\partial \mathbf{n}}$

$$\frac{1}{2} \frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}) + \int_{\Gamma} \left(\frac{\partial \Phi(\mathbf{x}, \mathbf{y})}{\partial \mathbf{n}(\mathbf{x})} + \mathrm{i}\eta \Phi(\mathbf{x}, \mathbf{y}) \right) \frac{\partial u}{\partial \mathbf{n}}(\mathbf{y}) \, \mathrm{d}s(\mathbf{y}) = f(\mathbf{x}), \quad \mathbf{x} \in \Gamma, \tag{4}$$

where $\frac{\partial u}{\partial \mathbf{n}} \in L^2(\Gamma)$ is the unknown boundary data, $\Phi(\mathbf{x}, \mathbf{y}) := \frac{i}{4} H_0^{(1)}(k|\mathbf{x} - \mathbf{y}|)$ is the fundamental solution of the Helmholtz equation in $2D$, $H_0^{(1)}$ is the Hankel function of first kind of order zero, $\mathbf{n}$ is the normal direction directed out of Ω, $f(\mathbf{x}) := \frac{\partial u^i(\mathbf{x})}{\partial \mathbf{n}} + i\eta u^i(\mathbf{x})$, and $\eta \in \mathbb{R} \backslash \{0\}$ is the coupling parameter. The total field throughout D is determined by

$$u(\mathbf{x}) = u^i(\mathbf{x}) - \int_{\Gamma} \Phi(\mathbf{x}, \mathbf{y}) \frac{\partial u}{\partial \mathbf{n}}(\mathbf{y}) ds(\mathbf{y}), \quad \mathbf{x} \in D.$$

In order to achieve an accurate solution of (1) for large k with reasonable computational cost, the oscillatory nature of the solution must be taken into account when designing the computational method. Recent research has focused on enriching the approximation space with highly oscillating functions, such as plane waves or Bessel functions, in order to accurately represent the scattered field. To examine this work very closely, we survey some existing literature where some schemes were proposed. A common approach is to include in the approximation space a set of plane waves propagating in multiple directions. In the boundary element context, this approach was attempted by de La Bourdonnaye [7] under the microlocal discretization method and by Perrey-Debain et al. [13, 14].The approximation space can also be enriched by including a single solution of the Helmholtz equation, the known incident field, to capture some of the oscillatory behavior of the solution. The idea behind this approach is to represent the solution of the integral equation by a linear combination of basis functions, these basis functions being the product of a slowly oscillating amplitude and an oscillatory phase factor. The slowly oscillating amplitude can be viewed as the ratio of the total field to the incident field and the oscillatory phase factor could be the known incident field. Physical optics predicts that this ratio approaches a constant in the illuminated region and zero in the shadow region as $k \to \infty$ [6]. This ratio can be approximated by conventional methods, employing piecewise polynomials. This idea was first considered by Abboud et al. [1] for two-dimensional smooth convex obstacles, where they used a Galerkin boundary element method on a uniform mesh to approximate the ratio of the scattered field to the incident field. Their numerical experiments suggest that the number of degrees of freedom needed to maintain accuracy needs to grow like $k^{1/3}$ in the high frequency limit. This is a dramatic improvement over the usual $\mathcal{O}(k)$ for conventional methods, though a rigorous analysis is not available in [1] to support the numerical experiments.

A closely related method to that in [1] was recently considered by Dominguez et al. [8] for a two-dimensional smooth obstacle. In [8] a p-version boundary element method is proposed, that is, different degree polynomials are used to approximate the slowly oscillating amplitude in the illuminated and transition zones, and the slowly oscillating amplitude is approximated by zero in the shadow region. A rigorous error analysis is made, with error bounds showing that the polynomial degree needs to increase proportionally to $k^{1/9}$ to maintain a prescribe level of accuracy as $k \to \infty$. This was confirmed by numerical experiments using a Galerkin scheme. In fact the numerical experiments of [8] demonstrate that a reasonable accuracy can be achieved

by keeping the number of degrees of freedom fixed as $k \to \infty$.

All of the schemes in [1, 8] can be expected to perform poorly for obstacles which have corners. For obstacles with corners there is, as well as the reflected field, a diffracted field due to the corners. These are not well represented in the ansatz proposed in [1, 8]. A scheme related to those in [1, 8], for obstacles with corners, has been proposed by Chandler-Wilde et al. [4, 5], Langdon et al. [9, 10] and Mokgolele et al. [11]. Chandler-Wilde and Langdon [4] proposed a novel Galerkin boundary element method to solve the problem of acoustic scattering by a sound soft straight line convex polygon.

In this paper we applied a collocation scheme on a uniform mesh to approximate the ratio of the scattered field to the incident field. We begin §2 by defining our boundary element method, here we explain the implementation of the collocation method on a smooth object, taking a circle as an example. In §3 we present some numerical results, the relative errors seem remain constant as wavenumber increases. Finally in §4 we present the conclusion.

2.   Boundary Element Method

We begin by parametrising (4) on the boundary of a circle, Figure 1. We employ a uniform
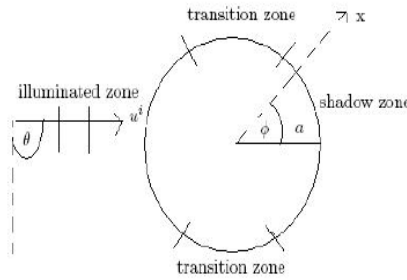
FIGURE 1. Scattering by a circle

mesh on the boundary of circle, and use a collocation method for computation of a numerical solution. The approximation space is a set of piecewise constants. As our starting point we re-write (4) in arc-length parametrised form as

$$(I + \mathcal{K})\psi(s) = F(s), \quad s \in [0, L], \tag{5}$$

where $\psi(s) := \frac{1}{k}\frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}(s))$, $L := 2\pi a$, $F(s) := \frac{2}{k}f(\mathbf{x}(s))$ and, for $v \in L^2[0, L]$,

$$\mathcal{K}v(s) := 2\int_0^L K(s, t)v(t)\,\mathrm{d}t,$$

and $K(s, t) = \left(\frac{\partial \Phi(\mathbf{x}(s), \mathbf{x}(t))}{\partial \mathbf{n}(\mathbf{x}(s))} + i\eta \Phi(\mathbf{x}(s), \mathbf{x}(t))\right)|\mathbf{x}'(t)|$ and for a circle of radius a

$$\mathbf{x}(t) = [a\cos t, a\sin t] \quad \text{and} \quad \mathbf{n}(\tau) = [\cos \tau, \sin \tau]. \tag{6}$$

Writing ψ_N as a linear combination of the basis functions gives

$$\psi(s) \approx \psi_N(s) = \sum_{j=1}^{M_N} v_j \rho_j(s), \tag{7}$$

where

$$\rho_j(s) := \left\{ \begin{array}{ll} 1, & s \in [0, L], \\ 0, & \text{elsewhere}, \end{array} \right.$$

are piecewise constant basis functions, M_N is the number of basis functions and v_j is the unknown to be determined. Substituting (7) into (5) gives

$$\sum_{j=1}^{M_N} \left[\rho_j(s) + \int_0^{2\pi} K(s,t)\rho_j(t)\mathrm{d}t \right] v_j = F(s). \tag{8}$$

The collocation method is to choose the sets of collocation points, $s_1, ..., s_{M_N} \in [0, 2\pi]$, and force (8) to hold at each collocation point, this leads to

$$\sum_{j=1}^{M_N} \left[\rho_j(s_m) + \int_0^{2\pi} K(s_m,t)\rho_j(t)\,\mathrm{d}t \right] v_j = F(s_m) \quad m = 1, ..., M_N, \tag{9}$$

Due to oscillatory of the kernel $K(s_m,t)$ and $F(s_m)$, the solution $\psi(s)$ will be oscillating everywhere between $[0, 2\pi]$ with larger amplitude in the illuminated region as wavenumber increases. In order to reduce these oscillations, some authors [3, 8] introduced the following ansatz:

$$\psi(s) := \frac{1}{k}\frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}(s)) = \psi_{\mathrm{slow}}(s)\mathrm{e}^{\mathrm{i}k\mathbf{x}(s)\cdot\mathbf{d}}. \tag{10}$$

Substitute (10) into (5) and dividing throughout by $\mathrm{e}^{\mathrm{i}k\mathbf{x}(s)\cdot\mathbf{d}}$ gives

$$(I' + \mathcal{K}')\psi'(s) = F'(s), \quad s \in [0, L], \tag{11}$$

where $\psi'(s) := \psi_{\mathrm{slow}}(s)$, $L := 2\pi a$, $F'(s) := F(s)\mathrm{e}^{-\mathrm{i}k\mathbf{x}(s)\cdot\mathbf{d}}$, I' is an identity operator and, for $v \in L^2[0, L]$,

$$\mathcal{K}'v(s) := 2\int_0^L K'(s,t)v(t)\,\mathrm{d}t,$$

where $K'(s,t) = K(s,t)\mathrm{e}^{\mathrm{i}k(\mathbf{x}(t)-\mathbf{x}(s))\cdot\mathbf{d}}$. Using (6) it is easy to evaluate the formulas for $F'(s)$ and $K'(s,t)$. Starting from (11), we can follow the same procedure as before to arrive to a system of the form (9). Precisely, we can define a boundary element approximation $\psi_{\mathrm{slow},N}(s)$, to $\psi_{\mathrm{slow}}(s)$, by

$$\psi_{\mathrm{slow}}(s) \approx \psi_{\mathrm{slow},N}(s) = \sum_{j=1}^{M_N} v_j\rho_j(s), \tag{12}$$

where the coefficients $v_1, \ldots, v_{M_N}$ are defined as the solution to a version of (9) with the functions K and F replaced by K' and F'.

3. Numerical results

For our numerical experiment, we code (11) on the boundary of circle, length $2\pi a$, where $a = 1$ is the radius of the circle. We evaluate relative L^2 errors $\left(\frac{\|\psi(s)-\psi_N(s)\|_2}{\|\psi(s)\|_2}\right)$ for fix $N = 8$. Here we take $\psi(s)$ to be our exact value, evaluated at $N = 16384$, (Note: is possible to derive the analytical solution of circle, for this particular problem). The relative errors seem to remain constant with increasing wavenumber k. These initial results demonstrate the robustness of our numerical scheme. In Figure 2, we show the total scattering field by a circle. This picture is infact the numerical results of Figure 1.

TABLE 1. Relative errors, Scattering by a circle

k	$\|\psi - \psi_N\|_2 / \|\psi\|_2$
5	$2.523 x 10^{-1}$
10	$2.770 x 10^{-1}$
20	$2.581 x 10^{-1}$
40	$1.9441 x 10^{-1}$
80	$2.025 x 10^{-1}$
160	$1.3970 x 10^{-1}$
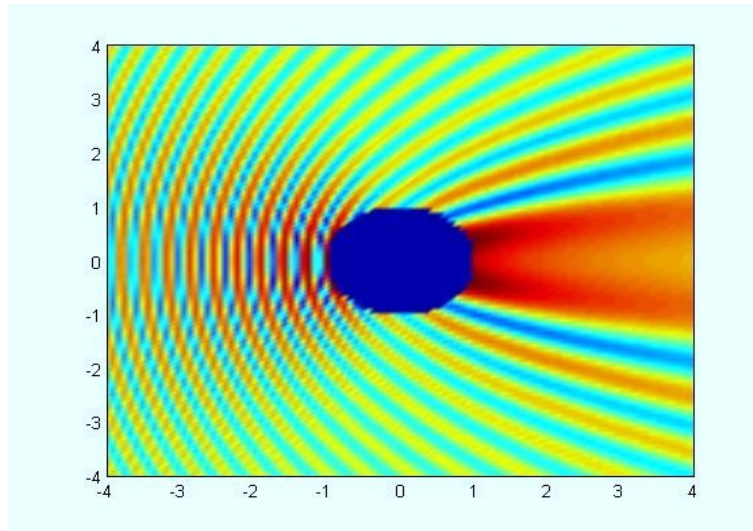
FIGURE 2. Total scattering field by a circle

4. Conclusion

In this paper we have proposed a collocation method for solving high frequency scattering by smooth objects. We demonstrated via numerical experiment that the relative errors remain constant as wavenumber increases, which in turn means, the number of degrees of freedom grows logarithmically with respect to frequency.

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Language practices involving two languages among trilingual undergraduate students of Mathematics

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Abstract

This paper presents language practices of some trilingual undergraduate students of mathematics as they engaged with a mathematics task. The paper draws from a larger study that was recently completed. The aim in this paper is to explore whether, how and why the trilingual students use languages in their repertoire to make sense of an algebra task. The two languages in focus are the home languages of the students and the Language of Learning and Teaching (LoLT), English. Research shows that there is a research gap on language practices in trilingual contexts. The study adapted a qualitative inquiry process. It was conducted in one public university with a focus on first year students undertaking mathematics in their programs. Data was collected using questionnaires, clinical and reflective interviews. Analysis followed Discourse analysis (Gee, 2005) with a focus on mathematical Discourses. Findings show that the students engaged with competent mathematical Discourses. Furthermore they used their home languages as resources in their repertoire to interpret and understand the task. There were multiple purposes for code switching between the two languages in their solitary engagement. The findings are important to inform Language in Education Policy (LiEP) in Kenya how and why some undergraduate students of mathematics position the home languages when they engage with mathematics. In the global perspective, the findings contribute to the field of mathematics education in trilingual contexts.

Language practices involving two languages among trilingual undergraduate students of Mathematics

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Abstract. This paper presents language practices of some trilingual undergraduate students of mathematics as they engaged with a mathematics task. The paper draws from a larger study that was recently completed. The aim in this paper is to explore whether, how and why the trilingual students use languages in their repertoire to make sense of an algebra task. The two languages in focus are the home languages of the students and the Language of Learning and Teaching (LoLT), English. Research shows that there is a research gap on language practices in trilingual contexts. The study adapted a qualitative inquiry process. It was conducted in one public university with a focus on first year students undertaking mathematics in their programs. Data was collected using questionnaires, clinical and reflective interviews. Analysis followed Discourse analysis (Gee, 2005) with a focus on mathematical Discourses. Findings show that the students engaged with competent mathematical Discourses. Furthermore they used their home languages as resources in their repertoire to interpret and understand the task. There were multiple purposes for code switching between the two languages in their solitary engagement. The findings are important to inform Language in Education Policy (LiEP) in Kenya how and why some undergraduate students of mathematics position the home languages when they engage with mathematics. In the global perspective, the findings contribute to the field of mathematics education in trilingual contexts.

Keywords. Trilingual, language practices, Discourse analysis, Home language.

1. Introduction

This paper presents some findings from a wider research study that explored language practices of some trilingual undergraduate students engaging with mathematics in Kenya. The paper focuses on whether, how and why some students draw on two languages of their three language facility to make sense of an algebra task.

When students engage with mathematics, a range of language practices within one language or multiple language environments may emerge. Furthermore, availability of more than one language in a classroom may open opportunities for the use of different languages for learning and/or teaching. Studies conducted in both bilingual and multilingual mathematics classrooms show how students draw on languages at their exposure to participate and improve their performance of mathematics. These studies have mainly focused on the Language of Learning and Teaching (LoLT) only (e.g. Barwell, 2003) or on two languages: the LoLT and students home

language (Adler, 1998; Setati, 2005). The LoLT in most cases is reportedly not the students home language particularly in studies conducted in Africa. In contexts such as Kenya, India, and Malawi, students also need to learn and gain fluency in a third language referred to as the national language. These languages are commonly used for communication between and among different language communities. The national languages, however, are not used in mathematics classrooms as LoLT (Chitera, 2009). In these contexts, the availability of the third language exposes the students to a third language in school and makes the students trilingual (see Hoffmann, 2001). Such trilingual students may also have other languages in their repertoire, but the concern of this paper is on three languages.

The languages are home language, the national language and the LoLT, here being English. They have different statuses and mathematics students are at one time or the other exposed to them in school. While research in mathematics education in bilingual and multilingual mathematics classrooms has recently been increasing in diversity and volume (Phakeng, 2013; Setati, Chitera and Essien, 2009), no research addresses how such trilingual students use their three languages while engaging with mathematics. This study will particularly deal with two languages of the three languages in the trilingual students language repertoire.

2. Why focus on tringualism?

[1] [2] Research on trilingualism is limited and as such there is no one definition of trilingualism that has been adopted (Hoffmann, 2001). Part of this limitation could be attributed to the assumptions about trilingualism: that it is an extension of bilingualism or it should be viewed as part of multilingualism (see Beaten Beardmore, 1982 in Hoffmann, 2001)). Furthermore there are implicit suggestions that several lingualisms can be subsumed under bilingualism or multilingualism. While the assumptions may be based on quantitative distinctions between the different lingualisms, research also shows that there are some common characteristics among bilingual and multilingual persons for example, the practice of code switching between languages (see Hoffmann, 2001).

Researchers in the area of trilingualism (Hoffmann, 2001; Ogechi, 2002)) accept the quantitative aspect of trilingualism. But there are also qualitative aspects that are characteristic of trilingualism (Hoffmann, 2001). First is that there different groups of trilingual, depending on both the circumstances and the social context under which they acquire and use the three languages. Second is that a trilingual speaker uniquely uses his/her three languages in ways that are determined by his/her communication needs. He/she has the ability to function like a monolingual, a bilingual or a trilingual depending on, for example, the topic, place or interlocutor. This requires a decision to code-switch. Trilingual people in fact assign, consciously or not, different functions to their three languages (Hoffmann, 2001).

Given these quantitative and qualitative aspects of a trilingual, it is observed that while a trilingual person may share some characteristics with a bilingual and/or multilingual person, a trilingual is not an extension of a bilingual but a special case of a multilingual person who retains characteristics of his/her own. From the foregoing discussion, a question that begs understanding is one on language practices of trilingual mathematics students when they engage with mathematics.

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[1]Home language: refers to the language commonly used at home and in the larger community. In some works it is also referred to as main language (e.g. Adler, 1998). A home language may be the first language of a speaker or other language acquired.

[2]This paper I have referred to a bilingual as to an individual who acquires and is proficient in two languages (Grosjean, 1982) and a multilingual as an individual who is proficient in more than two languages (see e.g. Chitera, 2009)

3. Background information and research in Kenya

Majority of Kenyan students become trilingual through schooling. During the first three years of their schooling, the Language in Education Policy (LiEP) stipulates that the predominant language in the school environment is used as the LoLT. The pre-dominant language is in most cases the home language of the majority students. During this period the students are introduced to formal learning of their home languages while Kiswahili and English are taught as subjects. Despite this trilingual facility, research in Kenya has been limited to the bilingual language facility of the students.

Research exploring the use of two languages in mathematics and sciences for instance, Kiswahili and English or Dholuo and English has been conducted particularly from the teaching perspective (Bunyi, 1997; Cleghorn, Merrit, and Abagi, 1989; Merrit, Cleghorn, Abagi, and Bunyi, 1992). However, there is no previous research in Kenya and elsewhere that is available that has dealt with how individual trilingual students, and in particular undergraduates, use their trilingual language facility when engaging with mathematics. In fact in her review of what has so far been researched in mathematics education and language diversity, Phakeng, (2013) shows that there remains a gap in research in trilingual contexts. In view of the aforementioned research gap, the current study is necessary. The study was guided by the following questions;

1. How do some trilingual undergraduate students in Kenya use their languages when solving mathematics tasks?
2. What language practices do these trilingual undergraduate students use when engaging with given mathematics tasks?
3. Why do these students use their languages as they do?

The questions above helped to focus on the students language practices through their verbal and non-verbal utterances, actions and reflections on their linguistic train of thoughts while they engaged with a mathematics task. The language practices of the trilingual students help to understand how the students position themselves as they engage with the task in relation to mathematics Discourse and in the use of the home languages and the LoLT and in turn how they position the languages as they engage with the task.

In the following section, I situate the problem of trilingualism in the wider field of mathematics education by drawing on literature in bi/multilingual contexts.

4. Situating the problem in the field of Mathematics Education

The language practices of trilingual students engaging with mathematics can be addressed through understanding the relationship between language and mathematics and the role that language plays in mathematics students performance. Furthermore, understanding language practices of bilingual and multilingual mathematics students can shed light into exploring language practices of trilingual mathematics students.

4.1. Language and Mathematics

One way of describing the relationship between languages, that is a natural language like English, and mathematics is in terms of linguistic notion of register (Pimm, 1987). Halliday (1975 in Pimm, 1987) argues that mathematics register has to do with how words and expressions are used in mathematics, styles of meaning and ways of arguing in mathematics. It can be developed in any natural language for instance, Kiswahili or English. In fact research (Halliday, 1974 in UNESCO, 1974; Pimm, 1987) shows that there are some defining characteristics of mathematics register in relation to the English language, hence the development of mathematical language. Part of knowing mathematics is acquiring control over mathematics register so as to be able to speak like a mathematician (Pimm, 1987). The student should therefore learn to use the language of mathematics and hence be able to construct, express and communicate the intended

mathematical meanings (see e.g. Pimm, 1981). As noted by Setati and Adler, (2000), the reality is that speaking, listening, reading and writing mathematics in multilingual classrooms requires the use of the LoLT, which students may not be fully fluent in. The foregoing discussion leads us to think of the ways of knowing and meaning in mathematics. In particular, it leads us to the formal and informal mathematics languages and how they are used in classrooms.

4.2. Formal and Informal mathematical language

Students come to school with everyday ways of knowing, speaking and writing mathematics which are different from the formal ways of using the mathematics register and hence the language of mathematics. According to Setati and Adler, (2000) formal mathematical language refers to the standard terminology or mathematics register which is developed within the formal settings like schools. It is the language valued in school mathematics. On the other hand, informal mathematical language is the kind that learners use in everyday life to express their mathematical understanding. Based on what they have acquired and how they manipulated meanings as young children learning how to mean, learners use their informal language in an attempt to assign meaning to unfamiliar mathematical phrases and expressions (Moschkovich, 2003). Although informal mathematics talks are inappropriate in formal mathematics settings, Moschkovich (ibid) observes that they should not be viewed as obstacles to learning rather they should be seen as valuable resources for developing learners mathematical competence. They can be used to assist learners in learning mathematics by moving from informal to formal language of mathematics that is valued in school mathematics. Hence the classroom plays host to both informal and formal ways of knowing, speaking and writing mathematics.

Moving from informal mathematics language to formal mathematics language involves learning mathematics within mathematics discourses. There are multiple and varied mathematics discourses. In what follows, I briefly discuss mathematical Discourses. To build on them, I start the discussion on Discourse Analysis.

4.3. Discourse Analysis and Mathematical Discourses

The discussion in this paper draws broadly on (Vygotsky, 1986, 1978) socio-cultural perspective and the analysis follows Discourse analysis (Gee, 2005) with a focus on mathematical Discourses (Moschkovich, 2002). Vygotskian socio-cultural perspective examines the roles of social and cultural processes as mediators of human activity and thought (Vygotsky, 1986, 1978). According to Vygotsky, (1986), human thought realizes itself, and is expressed, in words. Furthermore, mediation of the interactional process of thought and word occurs through culturally constructed artefacts which include elaborate sign system such as language. Language then expresses thoughts through verbal and non-verbal communication. The expressions can be found in the interactions within and between human beings, and between human beings and objects. Therefore according to Vygotsky, language mediates human thinking and social interactions, that is, communication within and between humans.

In his work of Discourse analysis, Gee (2005) acknowledges language as a tool for communication and also provides a method for analysis. Gee (ibid) sees the primary functions of language as to support performance of social activities and identities and, human affiliation within cultures, social groups, and institutions, hence Discourse (with capital D) analysis. Gee, (2005) makes a distinction between the terms discourse with little d and big D. He refers to discourse (with little d) as the language-in-use or the use of language on site (p. 7) and to Discourse (with big D) as language plus other non-language stuff. Thus he states Discourse as: ways of combining and integrating language, actions, interactions, ways of thinking, believing, valuing, and using various symbols, tools, and objects to enact a particular sort of socially recognizable identity (these are social practices) (pp. 21). From the quote above, we use spoken or written language in tandem with non-language stuff to perform actions in the world and hence project ourselves as certain kinds of persons engaged in certain kinds of activities (Gee, 2005). In other words,

language supports how we act as certain identities engaged in certain activities, and hence we get associated with groups whose members act as we do. Central to Discourse analysis is recognition; if the activity and identity are recognised, then one will have pulled off a Discourse of a sort (Gee, 2005, 23). The meanings derived in any one Discourse situation are multiple, varied and situated in context of use. In this paper, I focus on Discourse (with capital D) analysis.

Moving from informal mathematics language to formal mathematics language means that students are involved in learning the mathematics register. The knowledge of mathematics register facilitates mathematical conversations in the classroom (Pimm, 1987). In order for students to acquire this register, it is imperative that teachers teach from the basis of conceptual curriculum so that students come to learn and understand the intended curriculum (Thompson and Thompson, 1994).

In conceptual curriculum the teacher provides and asks students for explanations in the process of solving a task (Thompson and Thompson, 1994). Alongside the conceptual orientation is the calculational orientation, which is viewed as composed of procedural steps of mathematical operations. The distinction between these two classroom orientations has been extended into calculational and conceptual discourses by Sfard, Nesher, Sreefland, Cobb, and Mason (1998). These authors refer to calculational discourses as the discussions in which the primary topic of conversation involves describing the specific steps that have been followed to arrive at a mathematical solution. In contrast, conceptual discourse involves reasons for calculating in particular ways. Integrating the calculational and conceptual discourses at play with Discourses from Gee (2005), it is in the interest of this study to ask; when students explain their mathematical understanding conceptually and/or calculationally, in what ways do they integrate and combine language and non-language stuff to communicate this understanding? Literature so far reviewed show that students need to gain control of the mathematics register and use the valued formal mathematics language to communicate their mathematical understanding. However, it is the mathematical Discourses within their communication that is most important in deriving meanings of their utterances.

Using a situated and socio-cultural perspective and the notion of Discourses , Moschkovich, (2002) observes that mathematical Discourses constitute ways of combining and integrating language with other non-language stuff, and ways of saying, doing and being in mathematics. In particular, practices that count as participation in competent mathematical Discourse include particular modes of argument such as precision, brevity, logical coherence, abstracting and generalising, imagining, visualising, making mathematical connections, assumptions and using mathematical representations, justifying, gesturing and predicting (Moschkovich, 2002).

To be recognised as competent in mathematics Discourse, trilingual mathematics students need to engage with mathematics tasks in ways that mathematics competent people do and in ways that are acceptable in the mathematics Discourse community. They need to have developed the acceptable practices within mathematics Discourse. They learn these practices through interaction with others for instance through interaction with teachers, books and with other tools like calculators and mathematical tables, non-tangibles things like formulae and discussion with peers. It is in such interactions that negotiations of meanings happen and probably different and multiple ways of doing mathematical tasks are found. With time and continued practice, and building and rebuilding mathematical arguments, students develop mathematical competence in spoken, written mathematics and ways of acting and being in mathematics Discourse. If they project the ways of being in mathematics that can be identified with the mathematical Discourse; they enact socially situated identities and activities within mathematical Discourse. In learning and teaching then trilingual students are exposed to a range of mathematics Discourses. In line with that, this paper will reveal a range of mathematical Discourses that the selected trilingual students draw on and enact as they engage with mathematics.

It is however, important to note that mathematical Discourses are mediated through natural languages, for example English and Setswana. With that regard, a discussion on language

practices involving bilingual and multilingual students is relevant for this discussion. The most common language practices in contexts of language diversity reported in literature is that of code switching.

4.4. Language practices of bilingual and multilingual students

Studies in bi/multilingual classrooms have shown that code switching between students home languages and LoLT can support their participation and performance in mathematics classroom. In a review of these studies, I find out how and why bi/multilingual learners switch to their home languages when they engage with mathematics tasks that are presented in LoLT. The findings provide insights into language practices of trilingual learners.

4.4.1. Code switching. [3] [4]
Code switching is the alternative use of two or more languages in an utterance or conversation in a more or less deliberate way (Baker, 1993; Grosjean, 1982). The alternation can involve a word, a phrase, a segment of a sentence, a sentence or several sentences. It is a common characteristic of bi/multilingual speech and there are communicative purposes for which it is used. While the works of Baker and Grosjean portray code switching as a verbal strategy, a corresponding non-verbal strategy of language switching has been proposed and used in research on mathematics education. Language switching refers to the use of two or more languages during solitary and/or mental arithmetic computation (Moschkovich, 2005). Language switching is here seen to mean switching between two languages in thinking through computations. In this study I choose to refer to all situations where students switch between languages in verbal conversations or in mental computations as code switching. However, the differences between the language skills of conversation and mental computations or thinking will be noted appropriately. In a single speech, code switching may serve different purposes in communication and the languages involved may have a range of functions. In what follows I discuss the purposes for which code switching has been used in bi/multilingual mathematics classrooms and its influence in the learning and teaching process. In so doing I highlight the how and why code switching is practiced.

4.4.2. Purposes for code switching in bilingual and multilingual mathematics classrooms.
Research shows that code switching between learners home language and LoLT supports learning and teaching in bi/multilingual classrooms (Clarkson, 2006; Cleghorn, Merrit, and Abagi, 1989; Merrit, Cleghorn, Abagi, and Bunyi, 1992; Moschkovich, 2005; Parvanehnezhad and Clarkson, 2008; Planas and Civil, 2008; Planas and Setati, 2009). Whenever it is used in the learning and teaching of mathematics, code switching is used for a range of purposes. Some of the purposes are discussed below with the functions of the language(s) embedded in the communicative process. Furthermore, most of the studies that give focus to code switching focused on learners who are still learning the LoLT and had limited ability in it. Exceptions to these are for example studies of Clarkson (2006) and Parvanehnezh and Clarkson, (2008) where some learners had high proficiency in both the LoLT and the respective home languages.

Translating from one language to another

Code switching is practiced with the purpose of translating from one language to another for a range of linguistic reasons (Kern, 1994). It can assume verbal, written and mental communication. Furthermore, translation is only possible when the message is first understood in the original language. Some reasons for which translation has been used in mathematics classrooms are;

a. To express words or phrases in the language that is more familiar than the other

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[3]The situated and socio-cultural perspective looks at the use of situational resources students use and ways that mathematics Discourses are relevant to the situation.
[4]Discourse Analysis from Gee, (1996)

It has been observed that bilingual mathematics students translate mathematics task content due to familiarity with certain words and numbers in home language, and not due to lack of knowledge of such words (Moschkovich, 2005; Parvanehnezhad and Clarkson, 2008). Studies in multilingual settings show that teachers commonly switch from the LoLT to learners home languages to make lesson content familiar for the learners (e.g. Cleghorn, Merrit, and Abagi, 1989). Cleghorn et al., (ibid) observed that the use of local and familiar words may have expanded students' awareness of word meaning and language differences, helping to develop their English competence while also fostering understanding of the concepts taught. This was necessary as the students were still developing proficiency in English as they learnt science in it. The discussion on students familiarity with home languages is important for the trilingual students in Kenya. Some of the students translated the task content mentally because they were more familiar with home languages than they were with the LoLT.

b. To emphasize a point or certain words

Translation is used to emphasize a point or words or phrases (Baker, 1993; Merrit, Cleghorn, Abagi, and Bunyi, 1992). In this case of translation words are substituted with words of another language with the aim of putting emphasis; however no explanations are given for the words (Merrit, Cleghorn, Abagi, and Bunyi, 1992). For instance, in their study, Merrit, Cleghorn, Abagi, and Bunyi, (1992) observed that while speaking of a container, a Standard eight teacher used the word mkebe, the Kiswahili equivalent word for container, similarly for tapeworm, he used jofi the Dholuo word for tape worm. In so doing the teacher emphasised specific objects without explaining their meanings. Translation of some English words to home languages was evidenced in this study.

c. Translating all the time

Some students translate text all the time with the ultimate goal of transforming the information into a more usable representation (Kern, 1994). In his study with Vietnamese-English bilingual students, Clarkson (2006) observed that the Vietnamese mathematics students translated problems from English to Vietnamese. The use of Vietnamese was associated with the assistance the students got from the parents who prominently used Vietnamese or from siblings who used English and Vietnamese. Consequently while in school, the students translated the problems into Vietnamese while reading and thinking through them. They then translated back to English to make the ideas compatible with the classroom language situation. While it is not clear whether the students in Clarksons study translated all the content, he noted that the students did not translate individual words to check for meaning. Translation of all content may, however, pose a challenge since some of the terms of the mathematics may either not be available in home languages or are not readily used (Setati, 1998). Furthermore, translation does not always work to the advantage of students (Kern, 1994), since if content is inaccurately translated it may lead to misconceptions. Trilingual students may translate a given task to their home languages all the time for a range of reasons for example to seek understanding or due to familiarity with home languages. Such languages have implications for how they engage with mathematics. Most of the research reviewed here, which focuses on the use of code switching for translation purposes was done with learners in their early or middle stages of learning their LoLT. In contrast all the students in this study were academically proficient in LoLT, despite that; they translated the task content from LoLT to their home languages.

Context of using language
Research shows that the contexts in which bi/multilingual learners find themselves may facilitate or constrain code switching (Cohen, 1995; Grosjean, 1982). Contexts involve both physical

environment and the presence or absence of other people. Research in mathematics classrooms provides evidence of influence of context in language use (Clarkson, 2006; Parvanehnezhad and Clarkson, 2008; Planas and Setati, 2009). For instance, bilingual learners switch between home language and LoLT while operating in their individual private world or in small groups where they share a home language (Clarkson, 2006; Planas and Setati, 2009). On the contrary, bilingual learners remain in the LoLT when they are organised in linguistically mixed groups (Planas and Civil, 2008; Planas and Setati, 2009). In the study by Planas and Setati (2009), the students in their study continued using the LoLT despite being prompted to use their home languages, probably viewing their home language as a language that is not valued in mathematics classroom learning. This constraint to code switching may be more pronounced in environments where learners are restricted by the LiEP.

Given that code switching may be prompted by contexts, it is important to identify and understand the contexts that facilitate or constrain trilingual undergraduate students to switch between languages when they engage with mathematics tasks.

In concluding this section on situating the problem of trilingualism, it can be seen that language and mathematics are complexly related. Expressing their relationship in terms of mathematics register involves expressions that go beyond words and structures to styles of meaning and modes of argument. Students need to gain control of the register in order to use it as mathematically competent people do. While they need to learn the formal and valued mathematics register in school, they report to school with informal ways of knowing, more so in different languages which are not valued in school. In deriving meanings of their mathematical utterances, what is most important is to consider the mathematical Discourses within their communication either in LoLT or home languages or both.

For trilingual speakers, switching between any two languages is an important speech strategy (Hoffmann, 2001), as discussed earlier. They use their three languages within their linguistic environment and communication needs. Hoffmann observed that in education, trilingual students use the LoLT more commonly in external communication, while other languages are used for inner functioning. In the light of Hoffmanns observations on trilingual speakers, I explore how and why some trilingual undergraduate students in Kenya use two of their three languages facility when they engage with mathematics.

In the following section I briefly summarise the theory and method of Discourse analysis, which I have used to analyse the data.

5. Theory and method of Discourse Analysis

5.1. The Theory

In his work on Discourse analysis, Gee, (2005)provides a theory and a method for studying how language is used to enact specific social activities and social identities. He provides tools of inquiry and strategies of applying them in analysing data. The tools that are important for my analysis are social languages, Discourses, situated meanings and Discourse models. The tools help us to ask questions about what he refers to as the seven building tasks (Gee, 2005, 11), that we build when we use language and to understand how language is used as it is used. He identifies the building tasks as significance, activity, identity, relationship, politics, connections and sign system and knowledge. Gee observes that whenever we use language, we build at least one of the seven building tasks in more or less routine ways, because of our cultural inclinations of doing things. In line with these building tasks, discourse analysts can ask questions about any piece of language-in-use connected to the building tasks (Gee, 2005, 11-19). Alongside the identity and relationships building tasks, pronouns can help us to recognise the identity and activities that a speaker is enacting. Pronouns are commonly used when people talk. They code and convey aspects of speakers' personal identity and group association (Rowland, 1999). The

commonly used pronouns in mathematics talk are I, you and we (Pimm, 1987; Rowland, 1999). In using the pronouns, the referent(s) may be clear for instance I referring to the speaker, and you to the audience (single or multiple), or they may overlap for instance, I and you can be used to refer to the speaker, while you and we can be used to detach the speaker from immediate reference and hence make a generalization. Rowland observes that the variations in the way speakers use pronouns can be associated with delicate shifts of social positioning of the speaker in relation to his/her audience, to own up to something as an individual (I) or as a group (we) or partially dissociate oneself (you). The shifts in social positioning noted by Rowland resonate with shifts in identities and relationships (Gee, 2005) discussed earlier and hence the more the reason that the pronouns be used alongside the two building tasks in the analysis. It should be noted that when analysing utterances involving the pronouns you and we, there is complexity in decoding the co-referential. Therefore there may be multiple meanings that can be derived. According to Gee, social languages and Discourses are primarily relevant to how people build identities and activities and recognize identities and activities that others are building around them. These tools help us to talk about, and thus construct and construe, the world. Situated meanings and Discourse models are the primary tools of inquiry that deal with intricacies of how language is used (Gee, 2005). Gee observes that these latter tools guide inquiry in regard to specific sorts of data and specific sorts of issues and questions. It is worthwhile noting that all the tools do not work independently but are integrated one with the other.

In using language, we enact certain Discourses in the same or different contexts (Gee, 2005). He argues that that Discourses can be understood by situating meanings of words in specific contexts of use. In order to make sense of the situated meanings, we need to select the patterns and sub-patterns that are relevant in a particular context. The guide to the selection of the patterns resides in Discourse models of a persons socio-cultural groups and the social practices and settings in which they are rooted (Gee, 2005). Discourse analysis can be used to analyse language practices within one or multiple language environment. In fact when trilingual students engage with mathematics, they may project themselves as members of mathematics Discourse community using one language or switching between languages as par their communication need. Therefore I have flexibly adapted the four tools of inquiry, so as to ask certain questions about the building tasks in an attempt to understand language practices of the trilingual students involved in this study.

5.2. Method of Discourse Analysis

Gee, (2005) observes that discourse analysts are interested in analysing situations in which language is used. Such situations involve the contexts in which building tasks take place. He refers to the situations as discourse situations.

Any piece of language, oral or written, used in a situational network is composed of a set of grammatical cues and clues (Gee, 2005). The cues and clues contribute differently to the seven tasks and they guide us in knowing which of the building tasks are being built. They are carried out all at once and together and they may be in the same or different social languages.

Trilingual students control many different social languages and switch among them in different contexts. They also mix social languages in complex ways for specific purposes. In fact they can mix or switch between different social languages that are drawn from different languages at the level of national languages such as English or Kiswahili or home languages. As a result, a range of cues and clues are evident in their social languages.

The cues and clues help to assemble here-and-now situated meanings through which the seven building tasks are accomplished (Gee, 2005). In turn the situated meanings activate certain Discourse models. Finally the social languages, situated meanings, Discourse models at play allow people to enact and recognize different Discourses at work. This study identified similar patterns of grammatical features that are indicative of particular kinds of social languages that the trilingual students uttered as they did and reflected on the mathematics task.

To identify the indicators of cues and clues in the social languages, I have adapted some of the 26 questions that Gee (2005, 110-113) has proposed and I added other details of language that appeared relevant to my analysis. I looked at the utterances that illuminated my research questions in the light of Discourse situations and then took the key words or phrases which I analysed for cues and clues. In general, I have structured my analysis into three main phases:

1. Identifying the grammatical cues and clues in social languages of students responses to the algebraic task and in their reflections.
2. Identifying some key word(s) or phrase(s) in the social languages and looking at the situated meanings that they had. To make sense of the situated meanings, I selected the patterns of features that were assembled in the particular context and that implicated the Discourse models at play.
3. Explaining the different Discourses that the students seemed to enact through the social languages, situated meanings and Discourse models.

6. THE STUDY

[5] The data analysed in this paper focuses on first year trilingual undergraduate students of mathematics of Procity University , a public university in Kenya. Data was collected using three instruments; questionnaires, clinical, and reflective interviews. A structured questionnaire (Kothari, 2009) was used to gather the baseline data, which was necessary for selecting interview participants. A sample of 15 students was selected. They had a range of home languages and they indicated that they used other languages apart from English while responding to mathematics tasks. They also indicated the language in which they preferred for the interviews. Clinical interviews (Keats, 1997; Minichiello, Aroni, Timewell, and Alexander, 1990) were used to establish how the students used languages and other non-language stuff in relation to the mathematics task that was given. The reflective interviews were used to identify, ascertain and confirm various actions and languages and that were used during the clinical interview most of which were not visible during the interviews. Semi-structured questions were used to enquire on some critical moments while the students were engaged with the task. The reflective interviews also provided data on how and why the participants used each language while processing the task, in speech or in writing or other non-verbal means. Both interviews were video recorded and transcribed. Field notes, copies of students worksheets and questionnaire details provided supplementary data. The transcripts formed the primary data for analysis.
The task that the learners responded to was adapted from the standardized Kenya Certificate of Secondary Education (KCSE) of 2010 and read as follows:

[**Q**] A hall can accommodate 600 chairs arranged in rows. Each row has the same number of chairs. The chairs are rearranged such that the number of rows is increased by 5 but the number of chairs per row is decreased by 6
(**a**) Find the original number of rows of chairs in the hall.
(**b**) After the re-arrangement 450 people were seated in the hall leaving the same number of empty chairs in each row. Calculate the number of empty chairs per row.

Data from transcripts of the clinical interviews is analysed. It is supplemented by data from reflective interviews and the questionnaires. The students who switched between English and home languages were six. Four of the students S8, S6, S15 and S10 engaged with the task by switching to translate the whole task while two of them S3 and S4 translated parts of the task. The home languages of these trilingual students are Kiswahili for S8 and S6, Kikuyu for S15 and Kikamba for S10, Luluhya for S3 and Dholuo for S4.

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[5]A pseudonym for the university

TABLE 1. Key questions posed during the interviews

Clinical interview question	Reflective interview questions
When you first read the question, what impression did you get? Or other question that was relevant based on point of concern.	1. When you first read the question, what impression did you get? Or other question that was relevant based on point of concern.
	2. The question was written in English, which other languages did you use as you engaged with the task?
	3. Which other languages do you use while engaging with mathematics either in discussion groups or alone?
	4. Why did you use the languages as you did?

In the analysis, I present detailed analysis of S8 and S3 while I briefly examine the responses of S6, S10, S15 and S4 in the interpretation of the task. The aim of the brief examination is to support the findings from the language practices of S8 and S3 and perhaps raise other pertinent issues in the use of the languages. Furthermore parts of language practices of these students that involved identifying and explaining cues and clues in the seven Discourse situations are typical of the language practices of S8 and S3 under the same Discourse situations. However, since how and why other languages were used was key to my study, the brief analysis focuses on the aspect of code switching.

In the light of my research questions, during the interview I posed some key questions to the students. From their responses, I extracted the utterances that illuminated responses to my questions. With a focus on how they made sense of the task, I have analysed their utterances in the clinical interviews when they were interpreting the task, which was supported by their corresponding utterances in the reflective interviews. The questions are outlined in the Table 1 below and the utterances are presented as extracts. Below I present the analysis and subsequent discussion of the findings. The findings are that the students engage with activities and enact identities that people in mathematics Discourse community engage with. Furthermore they switched mentally between the LoLT and their home languages when engaging with the task to translate the whole task or parts of it for a range of reasons. Prior to the analysis, I present brief language and academic background of each student.

7. Analysis and findings

7.1. Translating the whole task

7.1.1. Analysis of S8's utterances in the interpretation. S8 was 20 years old student at the time of data collection. His first language is Kikuyu and the language he commonly uses at home is Kiswahili. He commonly used English and Kiswahili with himself, peers and lecturers. He scored a Grade A in mathematics and Grade B- in English. At the time of data collection he was undertaking a Bachelor's degree in Mechatronics Engineering. S8 switched between Kiswahili and English in interpreting the task. While this was not evident in his verbal and written explanation, he explained during reflective interview that he switched to Kiswahili in

thinking throughout the task.

Identifying and explaining cues and clues in the interpretation
S8 first worked on the task silently and then proceeded to explain from his workings. Below I analyze his response to what does the question require of you?

Extract 1
S8: This question, because here we have unknowns, we have kind of like before the rearrangement we know that the hall had a certain number of rows and each row had a certain number of chairs. So you give the number of rows an arbitrary letter like a like I have done here *(pointing where he had written)* a. I have said let the number of rows be a then, before the rearrangement, then I've said that because after the rearrangement the number of rows are increased by 5, so after the rearrangement the number of rows will be $(a + 5)$

Activity:

S8 was involved in making assumptions and formulating an expression and justifying both activities. Making assumptions, formulating and justifying are practices that are valued in mathematics Discourse (Moschkovich, 2002). Making assumptions was evident when he said "give the number of rows an arbitrary letter like a". He justified making the assumptions with the argument that since there were unknowns, then arbitrary values have to be assigned to these unknowns in an attempt to solve for the rows and chairs. Similarly S8 explained how he formulated the expression $(a + 5)$ and justified the formulation. Therefore S8 is recognized as a student who is involved in the activities of making assumptions, formulating and justifying the steps he took.

Identity:

In his explanation, S8 referred to *we*, later to *you* and in direct reference to his written work, he used the pronoun *I*. These pronouns are commonly used in mathematics talk (Rowland, 1999; Pimm, 1987). From the extract, his use of *we* and *you* suggest shared understanding of the conditions set out in the task and the unknown variables. In using these impersonal pronouns in formal mathematics language, S8 engaged with the task the way mathematicians do. Later he moved on to explain how to make the assumptions from his own perspective, which was based on his earlier written work. In using *I* he shows knowledge and ownership of the interpretation process again using formal mathematics language. His use of the pronouns is in line with mathematics register which researchers (Pimm, 1987; Rowland, 1999) argue that it is not entirely impersonal. Alongside using the pronouns, S8 is presenting himself as a mathematician, which is evident in the words he uses and how he approaches the problem. Choosing an arbitrary letter to represent a number which is not known is what mathematicians do. Formulating and justifying expressions to define or describe a situation are also some of the things that mathematicians do. S8 is recognized as assuming the identity of a mathematician at once in a general and a personal perspective.

Relationship:

The use of the pronouns *we* and *you* shows that S8 shares his understanding of the task like other mathematicians do. Later he uses *I* while explaining how the task should be done from his point of view. In doing so, he relates with other mathematicians as a student who ably approaches the task as other mathematicians do while at the same time uses words and modes of argument

as used in mathematics register and hence formulates the necessary mathematical expressions. Hence his relationship with the other mathematicians changes from a general relationship to a personal one.

Politics:

According to the first year mathematics syllabus of Procity University, students are expected to formulate and solve quadratic equations. Therefore formulating and solving quadratic equations in the context of the syllabus are social goods. These social goods were worth having in solving the algebraic task that was at hand. S8 made assumptions using arbitrary values and then formulated a mathematical expression. In making the assumption, he used formal mathematical language as used in mathematics Discourse. He then used the arbitrary values to formulate the mathematical expressions. Therefore S8 can be recognized as a student who distributed social goods across the task using formal mathematical language within mathematics Discourse.

Connections:

S8 is connecting his arguments with conceptual mathematics discourse (see Sfard, Nesher, Sreefland, Cobb, and Mason, 1998). This is evident in the way he connected unknown values of rows and chairs with some arbitrary values within the task which enabled him to formulate the required mathematical expressions. S8 described the assignment of arbitrary values to the unknowns and explicitly justified why he did so. By justifying, he connected his conversation to conceptual mathematics discourse. S8 is thus recognized as a student who made connections within the task, with conceptual mathematics discourse.

Sign systems and knowledge:

Throughout the Extract 1, S8 used formal mathematics language (in English) as noted in the discussions above. For example in assigning arbitrary values to the unknown number of rows and chairs he said So you give the number of rows an arbitrary letter like a. While this language is acceptable in mathematics discourse, it did not involve words commonly used in making mathematical assumptions, but it is clear that he was involved in making an assumption. Therefore S8 privileged using formal mathematics language in English.

S8 uses English language only in his verbal explanation. With the awareness that he used Kiswahili and English when working on mathematics when on his own, with peers and lecturers and the fact that he also uses Kikuyu at home, I inquired about his use of other languages in this particular task during the reflective interview. He responded as follows:

Extract 2
S8: Kiswahili only · · · even I found the answer using Kiswahili because now I was reading the question and interpreting it into Kiswahili and even these things I was writing I was saying wacha hii ikuwe hivi [Let this one be like this] referring to scribbles on arbitrary values on the question paper so in Kiswahili.
R: Why did Kiswahili come into play?
S8: Because I'm mostly acquainted to Kiswahili as a language; I'm best in Kiswahili than in English. So it was the language that I was using to interpret now this. So, even though it's written in English here, whatever was coming from my mind was in Kiswahili. I was just writing in English because it's like a requirement.

S8 said that he used "Kiswahili only", meaning that he did not switch to Kikuyu in the particular task. He read the question as it was and interpreted it into Kiswahili. By listening and observing him interpreting the task, I could not tell that the arbitrary values had been thought of in Kiswahili since these were all written in English in his worksheet. In using languages that way, he privileged Kiswahili, his home language, in interpreting the task. [6] S8s use of Kiswahili was based on the fact that he was more familiar with Kiswahili than English. This resonates with research findings (see e.g. Parvanehnezhad and Clarkson, 2008) that show that some bilingual students switch between LoLT and home language because they are more familiar with the home language than the LoLT. Furthermore S8 said that he is better in Kiswahili than in English. Being better in Kiswahili than in English could probably mean fluency in Kiswahili since Kiswahili was his home language, he used it at home and with the larger community. This explains partly why S8 used Kiswahili more than Kikuyu. Furthermore he commonly used English and Kiswahili with himself, peers and lecturers.

The fact that he only wrote in English was because "it was like a requirement". This positions English as a powerful language in S8's communication of the task. With such power of the language, he communicated in English whereas he would have done the same in his more familiar language of Kiswahili. After all English is the LoLT at Procity University. The perspective of English holding more power than Kiswahili in S8's case is similar to the view that English is the language through which one can have access to social goods like employment and language of examination (Setati, 2008). Since initially he had a choice of the language he preferred to be interviewed, in my view S8 restricted himself to using English in his communication with the researcher due to the power that English held.

In essence, S8 translated the whole task from English to Kiswahili solved it and then wrote in English. The two languages had different functions; Kiswahili was a language for internal communication that he used for thinking and interpreting the task, while English was for external communication, in writing and verbal communication with the researcher. He switched to Kiswahili for the purpose of translating the task due to familiarity with the language. Therefore code switching supported S8's participation in competent mathematics Discourses.

Discussion on S8's interpretation

From the analysis above, S8 made formal mathematics language significant. He used it in describing, formulating, making assumptions and justifying his workings. He at times positioned himself as working from a personal point of view and at another as working with other mathematicians. He used language to make connections within the task and with mathematics register, formal mathematics language, conceptual mathematics discourse and mathematics Discourse practices. In all his verbal and written explanation, he used a formal mathematics language in English and in thinking he used Kiswahili. He used Kiswahili because he was more familiar with it than with English. However, he wrote his work in English because the context required so, that is it was the expected language in such situations. Since initially he had been given the option of using any language in his repertoire, I argue that by using English in his reporting while he engaged the task in Kiswahili, he positioned English as the more powerful language than Kiswahili.

Therefore S8's utterances in interpreting and solving the task contain cues and clues for mental code switching, formal mathematics language, and conceptual mathematics discourse, and general and personal relationship with other mathematicians and in general mathematical Discourse practices.

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[6]This contrast of performance in Kiswahili and English was not based on any available academic records.

7.1.2. S8 Negotiating the situated meaning of key word "hall". S8 situated the meaning of a hall from the perspective of an auditorium or a theatre. He explained his experience with theatres in high school, as elevated from front to back. S8 had experience with theatres in that as a student of English and Kiswahili literature in high school he attended performances of a range of English and Kiswahili literature set books in theatres. He therefore was operating with the Discourse model of a hall as an auditorium or a theatre.

7.1.3. Emerging Discourses. From the discussion on the cues and clues in S8's language practices in interpreting the task and situated meaning and accompanying Discourse model, S8 "pulled off" a Discourse of a student who combined and integrated verbal words with formal mathematics language, and conceptual mathematics discourse, mathematics register, and mathematics Discourse practices. He did so by switching from English to Kiswahili to translate the whole task in his thinking to interpret the task. He assumed both general and personal perspectives of a mathematician whose view of the hall in question was an auditorium or a theatre.

In the next section, I analyse in brief how S6, S10 and S15 used their two languages as they interpreted the task. The analysis supports the findings from the analysis of S8 utterances and raises other pertinent issues on language practices in the interpretation of the task.

7.1.4. Language practices of S6, S10 and S15 in the interpretation. S6 was 21 years old Mechatronics Engineering whose first language is Kikuyu and his home language is Kiswahili. He explained what the task required of him in both verbal and written forms in English language only. To be able to do this, he translated the whole task to Kiswahili in an effort to understand the task better. This was because he had a better understanding of Kiswahili than English. S10 was a Civil Engineering student aged 19 years. His home language is Kikamba. He rarely used either Kiswahili or English at home. All the three languages were relevant to him when engaging with mathematics such that he used all of them when consulting with himself and when discussing with his peers. With the lecturers he used Kiswahili and English. In working on the task, S10 used English throughout his written and spoken explanation of the interpretation. In his reflections, he revealed that he translated the whole task to Kikamba in an attempt to interpret it, because Kikamba was the more familiar language. Below are his reflections.

Extract 3
R ... Did other languages come anywhere in between when you were responding to the question?
S10: Yeah, yeah, yeah. First after seeing the question, in all my studies, I try to interpret in Kikamba, which I'm more conversant with. I read in English then I interpret it in Kikamba, which I can understand more than English.
R: Are there particular parts or it is the whole question that · · · .
S10: The whole question.
R: How do you put it in Kikamba?
S10: I do it in Kikamba then I transfer to the paper in English.
R: Is it *(translation)* something that you can write?
S10 No, no, no. Yeah, I'm more conversant with Kikamba more than any other language.

The Extract 3 shows that S10 not only translated the task that was at hand but he did so all the time in other tasks. He does so because he is more familiar with Kikamba than English. He interpreted the task to Kikamba in his mind and neither verbalised it nor wrote it in Kikamba. It is interesting that when requested to write the interpretation in Kikamba, S10 said an emphatic 'no' while still arguing that he is more conversant with the language more than any other. While this explanation may seem to contradict his use of Kikamba, it in a way shows that conversational proficiency in a language is not commensurate with written proficiency (see Gerber, Engelbretch, Harding, and Rogan, 2005). He used English to communicate the task expectation and interpretation with the researcher. Therefore, Kikamba his home language was

significant for understanding and interpreting the task while English served to communicate with the researcher.

Similarly S15 read the task in English then interpreted the whole task internally in his home language, Kikuyu, for better understanding and he wrote out the solution process in English. This is a practice that he engaged with even in other tasks. Similar to S10's position, S15 he could not write the translation in Kikuyu though he could speak the language.

7.1.5. Discussion on translating the whole task. The language practices of the trilingual students S8, S6, S10 and S15 above show that they mentally translated the whole task to home languages in interpreting the task because they were more familiar with the languages, then they were in English. This finding supports Hoffmann's finding that some trilingual students use LoLT for external communication while their other languages are used for internal communication (2001). Furthermore the students translated tasks at all times similar to how English-French bilingual students mentally translated French text to English (see Kern, 1994). Their translations are associated for the common use of the languages as home languages for communication within their home environments (S8, S6, S10, and S15) and in mathematics group discussion (S8, S6 and S10). The fact that their switched to their home languages because they were the languages of common communication at home, are similar to the reason why English-Vietnamese bilingual students in Australia switched between their two languages (see Clarkson, 2006).

Setati, (1998) observes that translation may pose challenges when everything is to be translated from one language to another. Setati observed that some of English mathematical terms may not be used or are readily not available in home languages. The students in this study translated the whole task and did not report any challenges in translating the task. The fact that the task that the students engaged in did not have technical terms could be reason that they did not report translation challenges from English to their home language and vice versa. Next I analyse the utterances of S3 and S4 who translated parts of the task into their respective home languages.

7.2. Parts of the task

7.2.1. Analysis of S3's utterances in the interpretation. S3 was aged 19 years at the time of data collection. His home language is Luluhya, the language he commonly uses with his family and with little use of English. In the questionnaire, he indicated that he communicated with himself and peers more in English than he does in Kiswahili, while he used both English and Kiswahili with lecturers. He was pursuing a degree in Geomatic Engineering and Geospatial Information Systems.

Identifying and explaining cues and clues in the interpretations

S3 explained and wrote his interpretation simultaneously. In responding to what the question required of him, he explained:

Extract 4

S3: so the total number of chairs *writing* total number of chairs is 600 then each row has the same number of chairs. If it is 20, twenty, twenty per row is the number. Reading from the question he continues, so you just let the original number, original number of rows before the increase of rows be a value let's say x. So after the increase the new number of rows is now $(x + 5)$, yes after the increase. But the number of chairs per row is decreased by, by 6. So if there are x rows initially and the chairs the total number of chairs are 600, it means that the number of chairs per row in this will be $600/x$

Activity:

S3 used a specific assumption about the number of chairs that is 20 chairs per row which seemed to lead to the general assumption on the number of chairs. The reason as to why he first used a specific assumption before coming up with the general assumption about the number of chairs does not seem obvious. It could probably be explained by his response to the question of the impression the task created in him, which he associated with a real life situation as he explained during the reflective interview;

Extract 5
S3: First of all, when I read the question I looked at it and I tried to relate it to real life situation. When I / I read it further, I imagined myself arranging that *Inaudible* may be somebody has been appointed to arrange chairs in a certain hall. I was imagining if it were me, what could I do?
R: The real life situation?
S3: It's like there is a meeting in a certain hall, and may be the guests have been invited and everybody. So the chairs are supposed to be arranged, may be I'm there and I can be consulted / to arrange those chairs. So I was imagining, I'm the one to be appointed to arrange those chairs, what could I do?
R: Did that *(real life situation)* influence how you solved the problem?
S3: Yes in that, I was now seeing things like physically not like on the paper, because when I was focusing, the chairs as objects. That really motivated me, helped me a lot in solving that question'.

S3 explained his view of the real life situation as one in which guests had been invited to a meeting in a hall and he was required to arrange the chairs for them. The chairs were physical objects that assisted him in solving the task. While this activity was not visible when he was interpreting the task, it was important in understanding how S3 came up with his assumptions. S3 is recognized as making assumptions based on a real life setting a practice that is common in mathematics Discourse. Identity: He refers to "you", "let" and "let's" in making an assumption on the number of chairs. "Let" is commonly used in formal mathematics language in making assumptions in the mathematics Discourse community and is a defining characteristic of mathematics register (see Halliday, 1974 in UNESCO, 1974). Using the three referents, S3 positioned himself as working with and like other mathematicians who share the knowledge of making assumptions. In so doing, he assumed a general identity of being one of the mathematicians. Relationship: Following the general identity of belonging to mathematics community, S3 created a general relationship with the community. Connections: S3 makes connection within the task, and with formal mathematics language and calculational discourse (Setati and Adler, 2000; Sfard, Nesher, Sreefland, Cobb, and Mason, 1998). This is evident in that he initially takes on a specific assumption of the number of chairs and then connects it to a general assumption of the number of rows and finally to the number of chairs. Further he describes the steps he follows using formal mathematics languages. Sign systems and knowledge: Throughout the task, S3 used English. In the questionnaire he had indicated that he engaged in mathematics in English and Kiswahili, to know whether this also applied in the given task, I enquired how he used other languages.

Extract 6

S3: *(Laughs)* in interpreting that question, yeah to some extent. For deeper understanding of the question, when I read it I tried to interpret it in my language which is Luhya . I tried to translate the words written there in Luhya using my brain.
R: Are there specific words that you may have used in Luhya?
S3: Yes / like chairs is 'izidindeve', 'Izidindeve' are arranged in rows meaning chairs arranged in rows, arrange is "kubanga'a". When I had these two words, I now know that, I had the deeper meaning of this question, because I understand this language better, it's my original language, the language that I learnt.

S3 explained that he translated words like chairs and arrangement mentally to emphasize their meaning (Merrit, Cleghorn, Abagi, and Bunyi, 1992). It was necessary for him to understand the words chairs as 'izidindeve' and arrangement as "kubang'a" so that he could experience a deeper meaning of what the task was all about. He translated these words into Luluhya because he understood the language better, as he put it; Luluhya was his original language, the language that he first learnt. His statement suggests that he strongly relates with Luluhya and understands it better than English and Kiswahili languages. It was the more familiar language and hence was most relevant in interpreting the task. However, the words were neither verbalised nor written while he was engaging with the task. While he understood Luluhya better than other languages, I wondered why he preferred to be interviewed in English. In his explanation, English is positioned as the official language that should be used in engagements such as the interviews.

Extract 7

R: Much as you understand the Luluhya language, you preferred to be interviewed in English.
S3: *(Laughs)* the language I thought that the English is the most official to be used in interviews.
R: Otherwise,
S3: if they could allow laughs for any other local language then I could choose on this Luluhya. [7]

From the utterances in Extract 7, S3 shows that he would have preferred to be interviewed in Luluhya if it was allowed. When he was given the opportunity to choose his preferred language for the interview, S3 chose English. This shows that he may have avoided using his home language because English was the official language of communication in such settings particularly at university level. His choice of English shows the tacit power that the English language held in S3's language practice. Furthermore, when he says that 'if they could allow laughs for any other local language then I could choose on this Luluhya', it shows that the allowance given by the researcher in the context of data collection was not enough encouragement for him to use his home language. Therefore S3 remained in English language for his verbal communication and switched to his home language in thinking due to the context in which he found himself at the time of the interview.

Discussion

It is observed that S3 made the required assumptions based on some real life situation. Relating a task to a real life situation is a common practice among mathematicians. He used a single variable to form simple equations. Furthermore, he used plural pronouns which suggested

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[7]Luhya language is also referred to as Luluhya

that he identified with and worked in collaboration with members of mathematics community. S3 switched to Luluhya in thinking to capture the essence of the words chairs and arrangement. He argued that when he translated the two words he gained deeper meaning of the task. He translated the words in order to express them in the language that he was more familiar with. In contrast, S3 was restricted from switching to his home language in his verbal communication by the interview context. He thought in such context, English was the most appropriate and official language. This is consistent with the situation in which bilingual students remained in the LoLT (Planas and Civil, 2008; Planas and Setati, 2009), he perhaps did not feel permitted enough, to use his home language, Luluhya. In fact like the bilinguals in the study of Planas and Setati, (2009) he probably did not view his language as valued in such context. S3's self-restriction to code switch left me wondering how else he would be convinced that he could switch between different languages as he communicated verbally in the given context. In his explanation, S3 makes connections with mathematics register, formal mathematics language and calculational discourse which suggest that he identified himself, and related, with mathematics Discourse community. S3's utterances contained cues and clues for mathematics register, formal mathematics language, calculational discourse, generality, and code switching during thinking, and mathematics Discourse practices.

7.2.2. Situated meaning of the words "chair" and "arrangement". According to S3, his understanding of the task capitalized on grasping the meaning of the words "chairs" and "arrangement" (see Extract 6). He situated the meaning of chairs and arrangement in a real life situation of a hall. He simulated himself arranging the chairs in the hall in which a meeting was scheduled which helped him to view the chairs in the task as physical objects (see Extract 5). It seems that it is in such an environment that he could talk about chairs as 'izidindeve' and to arrange as 'kubang'a'. Thus he situated the meanings of chairs and arrangement relative to his socio-culturally defined experiences with how he could arrange the chairs in such a familiar setting. He applied the situated meanings against a Discourse model of a hall in his familiar environment and the socio-cultural practices involved in such an environment to which he belongs.

7.2.3. Emerging Discourses. From the preceding discussion on how S3 used language, he is recognised as having pulled off a Discourse of combining and integrating English and his home language in interpreting the task, mentally translating key words of the task into his home language for the sake of gaining deeper understanding. Further he combined and integrated formal mathematics language, mathematics register, calculational discourse and mathematics Discourse practices and he identified himself with mathematics community. He also operated in the Discourse of a student who viewed the use of other languages in their communication with the researcher as restricted. In the next section I analyse how S4, used languages when he interpreted the task. The brief analysis focuses only on code switching between English and his home language Dholuo. The analysis supports the findings from the analysis of S3 utterances.

7.2.4. Language practices of S4. S4 commonly spoke Dholuo at home than either English or Kiswahili, in fact Dholuo was his home language. When doing mathematics alone and with peers he did it in English and Kiswahili while he used English with the lecturers. He was enrolled for a degree in Geomatic Engineering and Geospatial Information Systems. In responding to what the task required of him, and in fact throughout the task, S4's verbal and written explanations were in English. I asked him during the reflective interview if, and if so when, he used Kiswahili and/or Dholuo in solving the task. He responded that he translated the task at the interpretation stage; in particular he translated the third sentence to Dholuo because he found it difficult to interpret English. In his words, Dholuo was the language that he was more familiar with and by switching to translate the third sentence; it was easier for him to understand the rearrangement and the task at large. Therefore S4 privileged to translate the part of the task he found difficult

into his home language in order to understand the task in a better way. I observed that when I asked S4 to say more on his translation, in his response he first asked whether he can say it in Dholuo to which I responded in the positive. S4's request to use Dholuo shows that it's uncommon to use other languages other than English especially in external communication at this level of education in Kenya. His request to verbalise and write in Dholuo resonates with S3's response when he said that if he was allowed he would have used Luluhya in the interviews.

7.2.5. Discussion on translating parts of the task. Students can switch between languages in order to express words or phrases in the more familiar language. S3 and S4 switched to Luluhya and Dholuo respectively because the languages were more familiar to them than the LoLT. At the same time they needed to gain better understanding of the task that was at hand. While S4 had challenges in understanding the third sentence and hence the translation, S3 needed further understanding beyond what he had captured in English.

8. Summary of findings on language practices involving the use of English and home language

The major findings of the case study reported here are that the language practices of the trilingual undergraduate students are identified with mathematics Discourse and with the use of two languages in their mathematics engagement. They participated in mathematics Discourse practices in a range of ways similar to those indicated by Moschkovich, (2002). While only the findings of S8 and S3 are presented in detail, in most other aspects, the language practices of the other four students were similar to those of S8 and S3. The language practices that show that the students used languages to engage with mathematical Discourses include: making assumptions, identifying variables, and formulating simple equations and justifying (or not) their workings in logical manner. Furthermore they constructed sentences that were rich in words and modes of argument used in mathematics register, formal mathematics language, and in mathematics Discourse. Their approach and ways of working as well as the use of pronouns positioned them as individuals and/or as members associated with mathematics Discourse community, thus assuming individual and/or as well as a general identity and relationship. It can be observed that the students had acquired control over mathematics register, they engaged with conceptual or calculational mathematics discourses and in general participated in mathematics Discourses switching between the two languages.

The language practices that emerged, involved English and home languages. The home languages were either first languages or Kiswahili. The first languages are however not taught or used at university level in Kenya while Kiswahili is taught as a subject. Despite that the home languages were resources for interpretation and understanding the task. For these students, code switching was necessary in making sense of the task. The purposes of code switching were translation and context in which they found themselves. Out of the six students, four students switched mentally to translate the whole task and did so all the time while the two others switched to translate parts of the task, to emphasize meanings. This latter case of translation is in some way similar to how bilingual Persian-English students translated certain words due to habitual use of the words in Persian language (Parvanehnezhad and Clarkson, 2008). They all switched to the home languages because they were more familiar with the languages than they were with English, the LoLT. The interview context restricted S8 and S3 from using their home languages, arguing that English was the appropriate language in such contexts.

The two languages had different functions: home language was for translating and exploring meaning mentally and English was for communicating verbally and in writing with the researcher. Furthermore S3, S4 and S8 were of the view that English was the language for formal communication and hence the need to use it when they engaged with the researcher. This

positioned English as the more powerful language of communication in such contexts of interview. Thus while the students home languages are not the LoLT in mathematics teaching and learning; they were resources for the students as they engaged with the algebra task. The trilingual students used two languages in their repertoire similar to how bilinguals students do (Moschkovich, 2002; Planas and Civil, 2008; Planas and Setati, 2009) and multilingual mathematics (Adler, 1998; Setati, 2005) and in fact in line with the observation by Hoffmann (2001) that trilinguals may use two languages in their reperrtoire as bilinguals do. Therefore the LoLT and the home languages that the trilingual students used in in verbal communication, writing and thinking in relation to their social culturally defined experiences, shaped the identities they enacted and activities they were engaged in within mathematics Discourse and in the use of their trilingual language facility.

9. CONTRIBUTIONS OF THE STUDY

The findings are important to inform LiEP in Kenya how the some undergraduate students position the home languages when they engage with mathematics. The students positioned their home languages as the dominant languages that facilitated interpretation, and understanding of the task, and English was dominant in the initial reading, verbal reporting and writing. The home languages were used in solitary and mental engagement with the task. The fact that they mentally engaged their mathematics in other languages was most unexpected because the students had learnt mathematics in English for the previous nine years and like in the study by Clarkson, (2006) one would have expected that they had learnt the LoLT sufficiently for them not to switch codes. Furthermore, their mathematics grades did not suggest that some of them could be struggling with understanding mathematics presented in English. The LoLT was used as a language of external communication while the home languages functioned as internal languages of communication. Thus while Kiswahili and home languages are not LoLT's beyond Standard three, the languages had a social value and worth in students interpretation and understanding of the task. Thus while English is the official and dominant language of teaching according to the LiEP, home languages played a significant role in students interpretation and understanding of the task. Furthermore, this finding has added to scholarly work (Clarkson, 2006; Parvanehnezhad and Clarkson, 2008) by establishing that code switching is not a reserve of students who are learning the LoLT rather code switching is a reality even for trilingual students who are competent in LoLT when they engage with mathematics.

In the global perspective, the findings contribute to the field of mathematics education in trilingual contexts. The fact that no research on language practices have been conducted among trilingual mathematics students (Phakeng, 2013) makes the findings of this study significant in field of mathematics education. As has been shown, the trilingual students in mathematics have specific ways of using their three languages (Hoffmann, 2001) which they have demonstrated in this study. That is they functioned like bilinguals. Therefore this study has provided insights on whether, how and why trilingual students use their languages as they do when engaging with mathematics tasks. Further since in this study trilingual speakers are considered as a special case of multilingual speakers, the findings of this study have helped to broaden the view of the language practices of multilingual students already investigated. Furthermore research on mathematics education in bi/multilingual contexts has commonly been concentrated at primary and secondary levels (see e.g Bunyi, 1997) with some exception in college level (Chitera, 2009). The study reported in this thesis was conducted a university level and probably the first research on language practices at university level. Findings of this study will give clues to mathematics lecturers of what transpires linguistically in their trilingual students as they engage with mathematics tasks.

10. Conclusion

In this paper, I have analysed language practices of six trilingual undergraduate students of mathematics who switched between two languages in engaging with a mathematics task. The languages were the LoLT and the respective home languages of the students. Using Discourse analysis (Gee, 2005), the language practices of the six trilingual undergraduate students have been analysed in an effort to communicate the socially situated identities and activities that students enacted as they engaged with a mathematics task. The analysis helped to understand whether, how and why the trilingual students' use language as they do when they engage with mathematics tasks.

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IMHOTEP

On Collaboration: An Important Skill for Mathematics Educators for the 21st Century

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Abstract

Mathematics, problem solving, and critical-thinking are key skills to innovation. Not only does the 21st century workforce require mathematics skills for success in everyday life but also for scientific advancement and technological development so as to enhance global competitiveness. Since teaching as an individual process in the 21st century classroom is no longer effective, mathematics educators should embrace practices that foster 21st century skills to their learners. Collaboration is a key skill that not only empowers teachers of mathematics in handling the bigger challenges of the 21st century education but also enables students to succeed in todays world. When we think about collaboration, many different types exist. Most of all major turning points in our history were motivated by a collaborative effort. With any new teacher entering the profession, one needs a mentor to help guide us through the first years of the profession. This is where collaborations start. However, collaborations in education should never stop and should always be on going, as there is always something new to learn. This presentation will outline why collaboration in education is important for both the mathematics educator and student. In addition, the presenters will also outline personal examples of collaborations and provide some ideas on how to obtain and maintain collaborators. This, we believe, can help prepare educators and learners for the challenges in life ahead.

On Collaboration: An Important Skill for Mathematics Educators for the 21st Century

Peter T. Olszewski and Dickson S. O. Owiti

Abstract. Mathematics, problem solving, and critical-thinking are key skills to innovation. Not only does the 21st century workforce require mathematics skills for success in everyday life but also for scientific advancement and technological development so as to enhance global competitiveness. Since teaching as an individual process in the 21st century classroom is no longer effective, mathematics educators should embrace practices that foster 21st century skills to their learners. Collaboration is a key skill that not only empowers teachers of mathematics in handling the bigger challenges of the 21st century education but also enables students to succeed in today's world. When we think about collaboration, many different types exist. Most of all major turning points in our history were motivated by a collaborative effort. With any new teacher entering the profession, one needs a mentor to help guide us through the first years of the profession. This is where collaborations start. However, collaborations in education should never stop and should always be on going, as there is always something new to learn. This presentation will outline why collaboration in education is important for both the mathematics educator and student. In addition, the presenters will also outline personal examples of collaborations and provide some ideas on how to obtain and maintain collaborators. This, we believe, can help prepare educators and learners for the challenges in life ahead.

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1. Introduction

Collaboration is a process of working together to reach a goal by putting talent and expertise to work. When we think about the word "collaboration," many different types exist. In the United States, most of the major turning points in our history were motivated by a group of people by a collaborative effort. These events such as the Declaration of Independence in 1776, the civil rights movement, and gay and lesbian rights would not have been possible if not for a select group of people who wanted change. In Kenya, the much enjoyed and celebrated independence would not have been possible if it were not for the collaboration of the famous Kapenguria six and a few more others.

In education, the days of having one teacher in the classroom with thirty or more students being the only primary source of where students learn are non-existent. The 21st century teacher is not simply coming to school, teaching, grading, talking to parents, and going home. The 21st century teacher is being inundated with seminars, conferences, individual education plans, emails from parents, good examination results and new teaching technologies to adapt to the new teaching practices for the millennial generation. Today's student too need to experience the environment they will enter as modern day workers and develop higher order thinking skills such as effective communication, collaboration, and being adept with using technology that they will need in the 21st century workplace. The 21st century skills therefore form an essential component of modern day learning and if properly inculcated into learning can reform it for the better. The educational practices of the traditional classroom are thus no longer effective and teachers must develop new teaching strategies that are radically different if success is to be realised. As Connelly and Clandinin (1994) point out: The horizons of our knowing shift and change as we awaken to new ways of "seeing" our world, to different ways of seeing ourselves in relation to each other and to the world. We begin to retell our stories with new insights, in new ways. *In this paper, we explore four key points:*

1. Why collaboration in education is important
2. The importance of collaboration for the 21st century educator and for the students
3. We will outline our personal examples of collaboration and what they mean to us, and
4. How to obtain and maintain collaborators.

2. Why is Collaboration Important?

Collaboration is the ability to work effectively with others, including those from diverse groups and with opposing points of view to achieve a common goal by solving problems, inventing, creating and producing results (Saxena, 2013). Now more than ever, two or more teachers and or students explicitly agreeing among themselves to meet and accomplish a particular goal or goals is a vital part of education (Clift, Johnson, Holland, & Veal, 1992). With the ever-changing world and the rate at which technology is growing, teachers need to have the latest and most up-to-date teaching practices and best practices at the ready to get students ready for global competitiveness. For Clandinin (1994), collaborative research projects are "attempts at establishing research relationships founded within conversations. Through these conversations, a mutual trust between the individuals is obtained with the openness and willingness to listen to other's point of view. Students need to get latest information and ideas from global perspective. Robinson and Darling-Hammond (1994) have identified 10 characteristics required for achieving successful school-university collaboration:

1. Mutual self-interest and common goals.
2. Mutual trust and respect.
3. Shared decision-making.
4. Clear focus.
5. Manageable agenda.
6. Commitment from top leadership.
7. Fiscal support.
8. Long-term commitment.
9. Dynamic nature.
10. Information sharing and communication.

We believe these are the ingredients to a successful and healthy collaboration in academia no matter the discipline. It is also important to note that collaboration builds strength among the participants. With the diverse knowledge from persons coming together, new ideas are formed from old ideas and practices. As Carse (1986) points out, "Power is never evident until

two or more elements are in opposition." Collaboration is a means to hearing a diverse body of knowledge and to build upon that knowledge to benefit not only the instructors but also the 21st century student.

3. Why is Collaboration in Education Important?

Collaboration is important because through it we can learn more and better by supporting each other emotionally and committing to cumulative efforts and effects. Studies and research are evident of the fact that better teaching and learning outcomes are achieved when teachers work in collaboration with each other. Working together encourages and contributes toward the achievement of a common goal.

Collaboration promotes further discovery in a research topic, classroom lessons, and provides new avenues of thinking and consideration that may have never been considered before. As similar to networking and meeting people at conferences, collaboration paves the way for ideas to flow among educators and students as well. It widens your circle of people you meet along your educational career and through a collaborative project, other ideas may formulate in terms of research, textbooks, and teaching practices. One can always improve and should always be willing to try new things. With the millennial generation, students are asking the question, "What will I use this for in life?" Simply teaching the pure mathematics engraves in our minds from our schooling years is NOT enough for our students. Students need to acquire 21st century skills for survival. Collaboration for the 21st century professors on how to teach students new skills they need and applications are at an all time high. They have a positive influence on student interest and engagement. Lesson study by Japanese teachers is one such gesture reported to have great impact on teaching and learning.

The importance of collaboration for the 21st century educator and the student According to Saxena (2013), the attitude of professional privacy is most prevalent among teachers of Mathematics. When it comes to their performance in the classroom, most teachers tend to restrict themselves and this hinders their growth, awareness and keeps them restricted to their conventional teaching practices and ideas. It should be realized that professional development and growth hugely depends upon effective teacher collaboration. Formal and informal collaborations help teachers learn a lot from each other and to get the most out of them teachers should seek each other, ask for advices and share how they work, they should spend time, help each other and build relationships. They should observe as many colleagues as possible to see how they teach or learn and seek out the ones they'd like to emulate. They should ask questions regarding student data, instruction, discipline and must be prepared to share their own knowledge. Systemic, critical and creative thinking skills which teachers gain through conversations and collaboration with colleagues should be encouraged so that the work they do with their students. They should bring up topics they want to discuss so that they can get ideas for strategies from each other. They can also analyze outcomes of what their students are doing either content-wise or may be 21st century skills as that gives them a chance to try things out in a collaborative setting before they work on them with their own students. Focus on skills transforms teachers from being mere knowledge givers to facilitators on how the relevant knowledge and skills can be acquired. This will help prepare our students for a future in which they become capable leaders of tomorrow.

4. Our personal example of collaboration what what they mean to us

In 2007, a collaboration between Prof. Dr. Kaisser of University of Hamburg in Germany and Owiti connected Owiti to Prof. Dr. Thomas Janke of the institute of Mathematics in the University of Potsdam in Germany who later invited Owiti to the institute and became a guide

in his PhD research and thesis writing. Besides, Dr. Bennard Bari, the academic director of AIMSSEC in South Africa, through Dr. Rejoyce Gadhi also extended an invite to Owiti in 2014 to take part in research and in service training programme for teachers of Mathematics. Owiti's most enriching collaboration is that with Prof. Olszewski (co presenter) in which apart from writing conference papers like this one and international journal publications, Owiti has benefitted by getting access to some of the latest publications and ideas on Mathematics Education. Owiti also have ties with Prof. Jill Adler of Wits which will lead to being more grounded on cutting edge research in Mathematics Education. A journey of miles starts with a few steps.

Our other personal examples of collaboration deal with Prof. Olszewski's first grant on the use of effective study skills using Exam Wrappers. In this grant, the Director for Student Initiatives teamed up with Prof. Olszewski to design question prompts to motivate students to use specific study skills for a Math of Money class. In addition, two of Penn State Behrend's tutors were brought in to share their perspectives on how they study. Prof. Olszewski and Prof. Daniel J. Galiffa established the installation of the 362nd Chapter of Pi Mu Epsilon at Penn State Behrend on October 12, 2012 through a collaborative and shared vision. Both men wanted to promote the mathematics excellence Penn State embodies in the mathematics major and recognize the students for their love of mathematics. Prof. Olszewski has also developed two courses at Behrend, the Mathematics of Money and a Hybrid College Algebra I class. Mathematics of Money was developed through the shared knowledge of several individuals at University part and the hybrid class was developed in conjunction with the nursing department as the class has specific focus for nursing students and with Jessica Resig, Director of the World Campus at Behrend. Lastly, Prof. Olszewski is currently piloting a study on online homework and if it can be used in a mathematics laboratory setting with instructor and tutoring assistance to help students understand concepts more effectively in College Algebra through PreCalculus.

5. How to obtain and keep collaborations

For a beginning teacher, this maybe a hard venture at the start. We offer the following suggestions:

a. Attend conferences and section NeXT meetings are full with other young faculty and seasoned faculty who are more than willing to pass down their knowledge to you. Always listen to others and ask questions at these meetings. In addition, be willing to travel to conferences like this one. The chances of meeting others in your same field are greater.

b. Be not afraid to talk to people no matter what their ranks maybe, this includes high school teachers. Sometimes, new faculty can be intimidated by higher ranked faculty but don't be. By asking them questions and advice, you may be more surprised how willing they maybe to offer some good advice. There is also a lot that can be learned about incoming freshmen by collaborating with high school teachers in your area. Take some time to reach out to your local high schools to get the inside perspective to the next incoming class for the fall semester.

c. Join the MAA or AMS or any Mathematical society. There are always many things one can do to help grow the amount of people you meet for possible common ideas.

d. Talk to your colleagues at your university as sometimes, one of your best collaborators maybe next door to your office. In short, get out of your office.

e. Keep conversations going on email, phone, etc. Have clear goals by making sure all people have a passion for the topic of interest, and be willing to change the course of your research and try different things.

f. Be ready for "no" answers too. Not all collaborations can happen and one must be willing to accept this.

   g. Write down your conversations, as ideas tend to leave the brain quickly when they are verbally talked about quickly.

There can be a lot more other strategies that schools can adopt to invite their teachers to collaborate with one another. Here are some strategies pertaining to real-life practices to help promote teacher collaboration:

(a) Video tape and discuss lessons: Teacher can videotape themselves while teaching a lesson and share those videos in small focus groups. The group members can watch the lesson and provide the teacher with feedback. This can give teachers time to have constructive conversations about the quality of instruction, teacher assignments, and student work. Consequently, discussions can lead to changes in instruction and student assessments.

(b) Make the most of interactions in Staffrooms: The staffroom is always the most appropriate place in a school for spontaneous interactions and sharing. Here, teachers find it comfortable to connect with each other, and the conversations they have often get them interested in what is being done in other classes, what teaching practices are being followed and how much progress the students are making. Participate in staffroom interactions to increase popularity of informal observations. This also encourages you to be part of other teacher's classrooms and observe their teaching which is a direct result of becoming interested in the discussions you have in the staffrooms about the teaching strategies. Teachers who are experts in different areas should not shy away from sharing their knowledge because it does not only benefit other teachers but also students.

(c) Encourage Professional Learning Communities (PLCs): Teachers should form voluntary regional collaborative groups in which they come together regularly to learn about and focus on particular topics. This will help teachers put what they learn into action, and to provide a comfortable environment where teachers are free to take professional risks. The first step in creating an effective learning community is to develop a shared vision, mission, and goal. If all teachers buy in, the collaboration will be meaningful and so provide more effective learning experiences for students. These communities can be developed with the help of a variety of resources such as websites, books, blogs, and videos that provide sample materials and information.

(d) Collaboration should be regular and scheduled: Experienced teachers should work closely with beginners to help them implement new curriculum and strategies and provide support and feedback. There could be scheduled weekly meetings during the school day in which teachers collaborate, set up observations, and provide feedback to one another. We hope these practices encourage teacher collaboration and you can adopt them in your own schools, colleges or universities to add a significant element to the professional development of and teaching practices by teachers.

Not all collaborations are a win, however, working with each other, we can always learn something new. Collaboration is not easy. Often times, there can be disagreements, arguments, and sometimes, feelings can be hurt. However, if we don't try new things and talk to fellow colleagues, how will we know what else is out there? How we will know if there is a better way to teach a simple concept as adding two unlike fractions? As Olson (1995) points out, "What is problematic to us provides the impetus for future learning as we try to understand ourselves and others, including the students in our case."

6. Conclusions

Many teachers today feel overwhelmed by the wide range of their students' learning needs and levels of preparedness. The educational practices of the traditional classroom are no longer effective and teachers must develop new teaching strategies that are radically different from

those employed in the traditional classrooms. The modern day classroom should be more centered on students and teachers should take the role of facilitators and guides instead of being mere providers of knowledge. They must ensure that they engage their students in learning and provide effective instruction using a variety of instructional methods and following different pedagogical approaches aided with technology. To achieve this they should be active participants in their own learning and must seek out professional development to improve their performance and their students' learning.

Teaching and learning in isolation are very restrictive and hinder progress. Learning in groups enhances the scope of learning and develops critical thinking. Collaborative learning redefines traditional student-teacher relationship in the classroom. Technology plays a big role in developing all of these characteristics for modern classrooms. These classrooms enhance the learning experience and better prepare students for higher education and workforce. We can accomplish more together than even the best of us can accomplish alone. For collaboration to produce results, group members should see each other as having different resources in the form of information, cognitive styles, cultures, decisions, etc., and should also understand that they have come together as equals to do a job. The problems we face today can be better approached through community effort because for complex problems or for a better understanding of any problem, multiple perspectives are required to analyze situations, imagine solutions and develop strategies to achieve them. What's most needed is the commitment that comes from collaborative work in order to follow through on our plans. Embrace the wisdom of crowds.

Collaboration is something that does not necessarily come naturally and working effectively with others is a challenging task in itself. Effectual collaboration requires training and the development of key personal skills. If teachers want their students to work together effectively they should explicitly teach and model collaboration skills. These skills include Active listening, Respect, Manners, Positive Attitude, being focused and Social Awareness. Simply telling students to work together wouldn't result in productive collaboration. Teachers should make students part of activities and projects where they find reasons to collaborate in order to accomplish a common goal. Students should be taught how they can be good and responsible group members through modelling, role playing, discussion, and facilitating. To foster the process of collaboration, teachers should adopt the terminology of teams rather than groups, since teams focus on accountability and commitment, are formed for a purpose and operate through norms and shared expectations. They should incorporate and adapt the high-performance principles common in a working environment to teams in the classroom. This requires time, good coaching skills, a focus on the quality of interaction between students, and a set of team tools, including contracts, rubrics, and exercise. Here are some ways in which collaboration can be taught and learned:

1. Assigning clear responsibilities to every member of the team and making clear what they need to accomplish as a team.
2. Providing illustrations of activities portraying collaborative work and efforts.
3. Assigning a leader who can responsibly take control of the activities of a group.
4. Encouraging self-direction and motivation.
5. Keeping track of the progress of the team and identifying areas of conflicts or their shortcomings as a team.
6. Conducting group and self-evaluations based on the progress.
7. Designing a rubric to measure the process and product.

To be part of the working world in the future, students need to learn to collaborate as a member of a team. Collaboration is one of the most significant skills required in a work environment and every student needs to be prepared for that environment, partly for employment opportunity, but mainly because learning and creating as an individual process is no longer effective. Effective communication is one of the main factors that drives collaboration and leads

to best expressions of innovation, creativity and critical inquiry. Once students get a little used to with working in teams, take responsibility and begin to collaborate well, they get to learn more, can assist and teach each other, develop powerful solutions and altogether enjoy the entire process if working in a team. This can lead to remarkable results. Teachers need to be involved with other teachers and mentors to learn about better teaching. They should come together, take a professional development and collaborate. In order for students to be successful and be able to demonstrate and apply these skills, teachers as adults in the learning system have to be ongoing adult learners and model the same skills in the work that they do professionally and in the classroom with the students. Being able to synthesize and transmit ideas in both written and oral formats is essential in collaboration. For teachers, effective communication and collaboration are the 21st century skills to be really emphasized on. Professional learning communities (PLC) which are effective at breaking the isolation of the individual teacher and help institute a collaborative community of practice should be formed. Focus on collaboration will help all educators to pause and reflect about what they do with the valuable time they have with students. There has been a lot of talk about knowledge economy and global competitiveness. The essence of these includes Creativity and Innovation, Critical Thinking and Problem Solving, Communication and Collaboration. These skills help develop the qualities that the students need to possess in the 21st century for success in college, careers and citizenship. Today's students are moving beyond the basics and embracing the super skills for the 21st century.

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The Science Behind a Flipped Classroom

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Abstract

When first asked to teach a flipped classroom, I was excited and apprehensive at the same time. I knew I always wanted to try it but I wasnt sure what the best practices were nor which parts of the class should be flipped and which shouldnt. I soon found out there is a lot more to flipping a classroom, but the results can be rewarding. In this paper, I will describe how I flipped a College Algebra class with a specific focus for nursing students. In addition, valuable ideas and best practices on how to effectively flip a class are presented.

The Science Behind a Flipped Classroom

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Abstract. When first asked to teach a flipped classroom, I was excited and apprehensive at the same time. I knew I always wanted to try it but I wasnt sure what the best practices were nor which parts of the class should be flipped and which shouldnt. I soon found out there is a lot more to flipping a classroom, but the results can be rewarding. In this paper, I will describe how I flipped a College Algebra class with a specific focus for nursing students. In addition, valuable ideas and best practices on how to effectively flip a class are presented.

Part 1. Math 21: Hybrid College Algebra I for Nursing Students

In the fall 2014, I was approached by my director asking if I would be interested in teaching a hybrid College Algebra course, which I gladly accepted. The course is part of the accelerated bachelor of nursing program at Penn State Behrend and I had to create a course with specific applications for nursing students. When first faced with this new endeavor, I knew I needed help from others. Reaching out to the various nursing instructors at Penn State Behrend, I gained a wealth of information on how to incorporate nursing applications into the class. When it came down to building the course online, I had help from my colleague, Jessica Resig, Director of World Campus Learning Design and Behrend Center for eLearning Initiatives. Through our collaborative efforts, the 7-week course was online and ready to go within a period of two months.

Part 2. Pre-Reading to Understand the Nursing Profession

When finding out this would be a class for nursing students, I wanted to find out, specifically, how mathematics is used in the medical profession. In the medical field, the administration of medicines is a large part of a nurses job. With the 21st century, the times nurses are preforming medical calculations have decreased, Hoiston (1996) estimated that at least one person dies each day in the United States due to medication errors. When first reading this, I strived to give my students all the mathematics they would need to be confident for their careers. In the nursing textbooks I have looked at, most of them present problems with formulas and having students substitute numbers without giving much thought about why they are calculating medical dosages. There are two large problems with this procedure. First, in a study done by

Hoyles, Noss, and Pozzi (2001) where twelve nurses were observed found the nurses mostly relay on proportional reasoning strategies. Although the nurses can remember formulas taught in their schooling years, they always returned to the methods of proportional reasoning to avoid any complex division and multiplication. The second problem with using the substitution method and formulas are outlined in the study by Blais and Bath (1992). Here, the conceptual errors in medication administration were more frequent than mathematical errors and measurement errors. The set-up of dosage calculations were of the more common type of error. Mistakes are made at any profession and are not unique to nurses. According to sociologist E. C. Hughes (1951), people working in all occupations make these errors and suffer a great deal of anxiety as a result. Wolf (1994) recalls how terrible she had felt when, Oh, my God! I pulled out the wrong one! The initial thought was to not tell anyone what happened, however, mistakes such as these in the nursing field must be reported. It is a matter of life and death.

Part 3. Outline of Math 21

As the course was part of the accelerated bachelor program for nurses, the nursing department wanted the course to be 7 weeks long. Typically, Math 21 is a traditional 15-week course that meets for 50 minutes on Monday, Wednesday, and Fridays or Tuesday and Thursdays for 75 minutes. Teaching this course in a regular fall and spring semesters is hard enough to get through material in a 15-week semester, let alone, a 7-week semester. I needed to think about how to best split up the class to make sure all bluebook concepts were covered along with the nursing applications. In terms of the required concepts, the course needs to cover concepts of quadratic equations, equations in quadratic form, word problems, graphing, algebraic fractions, negative and rational exponents, and radicals. The class would meet every Thursday for 2.5 hours, so time was critical to say the least. With the nursing applications I received from the nursing department, I wanted to include topics on unit conversions, Bishops Score, Radon Gas Risks, Growth Charts for Boys aged 2 20 years, Girls Growth Charts from birth to 36 months old, Fahrenheit vs. Celsius, Incidence of Diabetes in Adults ages 45 64, Life Expectancy, Training Heart Rate Zones, BMI, Calculating pH, Spread of Viruses, Cancer Cell Simulation, and Eliminating Medicine from the Bloodstream. As we can see, I had a lot of good topics to work with. Please see the appendix for the outline of Math 21.

During Week I, I showed students a PowerPoint on Scientific Notation, Linear Inequalities, Compound Inequalities, Absolute Value Equations, and Absolute Value Inequalities. However, for all other weeks, students were required to view video lectures I had created under the guidance of my colleague Dr. Jessica Resig.

Part 4. Course Development

By renting an iPad from the Behrend library, I created all videos using Doceri. When starting this process, I soon found out this was going to be a time consuming process. What saved me some time were PowerPoints I had created from previous times I had taught the course. Taking out the solutions to each problem, saving and exporting the slides as jpegs via Dropbox, I would have the problem typed up and I went through the solutions for my lectures. Some of the lessons I learned about making video lectures were:

1. Timing consuming but once made, videos can be used over and over again.
2. Record videos in a place where no one will disrupt you.
3. I recorded my videos between 6 11 pm each day in my office. However, make sure all phones are disconnected.
4. Dont create long videos. Studies show, students are more inclined to view 5 7 minute videos instead of 15 20 minute videos.
5. Use software where you can edit mistakes.

6. Have a script ready. This will minimize errors.

Math 21 was being offered in the spring 2015 semester. I started and completed making all videos within a three-month period between October to December. In terms of homework and assessments, I used MyMathLab (MML). I had used MML in the past at other Universities before coming to Behrend and I knew how successful the software had been in previous courses. Using the book by Martin-Gay, Intermediate Algebra 6th edition, Pearson had a Ready-To-Go course available with the book. All assignments were premade with the option to edit. I went through and edited each assignment by cutting out and adding questions. I also set the requirement that all assignments have a passing rate of 80When it came down to face-to-face class time, I started each class with a check-up activity to make sure students watched the videos before coming to class. Since the students were in the accelerated nursing program, they were ready to answer questions and actively participated in discussions. After the check-ups, the applications were passed out and students worked in pairs. The class had a total of seven students so there were a lot of interactions between fellow classmates and myself. This environment of students working in their own zones of proximal development where ideas were exchanged worked exceptually well for this class. If any work was not completed, students were given the opportunity to turn in any work the following week. I also accepted work until the end of the semester. My experience with teaching this course has been very eye opening and an experience Im thankful I obtained in my teaching career. I will now share some best practices I have learned that can take any traditional course to a flipped class. Of course, readers of this paper may have other ideas on how to effectively flip a class.

Part 5. The Science to Flipping a Class

When first asked to teach a hybrid class, I needed to learn what makes a hybrid class a hybrid class. According to Lorenzetti (2013), Instead of using class time to convey the basic information you want your students to remember and asking them to work on more difficult learning tasks alone, a flipped class asks students to come to class prepared with the foundational information and then to work on the challenging tasks of analysis, evaluation, and creation with others. When first starting to think about how to flip a class, which I had never previously done in my career, I had to think back to how college works. While going through my college career, I had to take my own initiative and learn things on my own. After a 50 minute class, I would spend 3-4 hours working out problems, re-working my notes, finding additional resources to make them work to aid my learning. Then, in class, I would ask follow-up questions to further engage myself to get the most out of my learning. What I have outlined above was my starting point to creating my hybrid class. The learning was placed on the students so that when they came into class, they were ready for the check-ups and to apply their knowledge to nursing applications. If we think about it, we as faculty spend many hours helping our students in office hours and answering emails. With a flipped class, the instructor can work with students directly as if they are in a big office hour. This, of course, will vary on class size. As Hill (2013) points out, Faculty can then devote time to helping students develop synthesis and explore application during class time through: experiential exercises, team projects, problem sets, and activities that previously have been assigned as independent homework. In particular, students can receive direct faculty input on those segments of the material that have historically been the most [difficult] or ambiguous. When teaching my flipped class, I found that topics such as factoring trinomials were easier for students to understand, especially when factoring trinomials where the leading coefficient was not one. With using colors on the video and having the check-up for this concept alone, I saw a great improvement in students being able to know what do to and to solve the problem faster than in a face-to-face class. However, as with teaching any class, the students must be driven to work. As Ullman (2013) points out, It requires students to be independent. Its an excellent growth opportunity, but the student has to be willing to put in the time and be

an active participant in the learning. Some [students] listen, do a little homework, and get by. That wont cut it in the flipped classroom. In a flipped classroom, it is imperative to convey to your students that the course is NOT a traditional face-to-face class. Caldarera (2013) writes, The flipped classroom format, [students] will be expected to complete the lesson as homework. All instructions are to be followed so as to allow more time for engaging enrichment activities in class. One important piece of advice I would like to share is to send out a welcoming email to the students before class begins. The email I sent out was the following: Dear Nurses, I would like to welcome you to Math 021: College Algebra I taught via the hybrid learning platform. My name is Professor Peter Olszewski and I will be your instructor for this class. I hope everyone is excited to start as I am. I wanted to send out a welcoming announcement about what we will be covering in the next seven weeks and my tips for success. Math 021 covers the following topics

1. Graphs, functions, and solving absolute value equations.
2. Polynomials and Factoring.
3. Rational Expressions, Equations, and Functions.
4. Radicals.
5. Quadratic Equations and Functions.
6. Sequences, Series, and the Binomial Theorem.

In terms of nursing applications, we will cover...

1. Unit conversions.
2. Bishop's score.
3. Radon gas risks.
4. Growth charts for boys 2 - 20 years & girls birth - 36 months.
5. Diabetes in adults ages 45 - 64.
6. Life expectancy.
7. Protein intake.
8. Heart rates.
9. BMI.
10. Spread of viruses.
11. Cancer cell growth.
12. Eliminating medicine from the bloodstream.

We will be using MyMathLab (MML) software for all homework assignments, quizzes, and exams. The start of each week will be on Thursdays and end the following Wednesday. My tips for success for this course and beyond are to....

1. RRR and SSS, which stands for Read, Read, Read, and Solve, Solve, Solve. The more you read, the examples, definitions, methods and the more you practice with solving the problems, the better you'll be! Practice makes perfect.
2. Be sure to take the time to read the information in the weekly assignments carefully. Everything you need to know is included in these areas.
3. Ask for help whether it is in the Problems and Solutions thread, weekly discussion threads, calling, or emailing me for help. Chances are, if you are stuck on a question, your fellow peers might also be stuck on the same question. Don't be afraid to ask!
4. Actively participate in ALL discussions. The more you chat about what you are thinking about to solve a problem, the better you'll understand it through comments from either myself or from fellow classmates.
5. Take your time when doing the exams. There are no extra bonus points awarded for blazing through exams in 15 minutes by getting half the answers wrong.
6. Check your work. If you get an answer to a problem, do it again to see if you get the same answer. If the problem requires you to solve an equation to get the value of a variable, substitute the value back into the equation to see if the equation still holds.

7. Get a calculator. If you don't already have one, a TI-83 or TI-84 would be good to have for future mathematics courses since they can graph functions. I know these calculators are pricy but they will be worth it in the long run. However, a simple calculator will work. I recommend the TI-30.

8. Buy some graph paper, a ruler, and pencils. These will help you with the graphing problems.

9. If you get frustrated on a problem, write out all the steps you took to solve the problem and either call or email me. By showing your work, I can quickly go through and see where you went astray.

It is great having all of you in class this semester. I hope you are looking forward to the course as much as I am!

Sincerely,

Professor Olszewski

In teaching this course, I have found that having this email and talking to students on day one about the course proved to be very helpful. Of course, the students I had for this course had taken other hybrid classes in the past and knew they needed to work hard. If you have a class of students who have never had a hybrid class they may be overwhelmed and even walkout on the first day. I have found that the student, who is driven to learn on their own, is willing to discover, and wants to be the best they can be, will thrive in a hybrid class. With my experience, I offer the following best practices to flipping your own class with the emphasis not all these ideas may work for every class and instructor:

1. **Make the class personal.** Often times, the only time your students will see you lecturing is through video lecturers. Try to make the videos, as you would want to be seen if you were being recorded for a giving a talk at a conference. Keep it professional but fun.

2. **Make it engaging.** When students are looking at your videos, if they find them boring, they will be more likely to resort to other videos to obtain information.

3. **Make it short and segmented.** As stated earlier, videos should not be long, as students will become disengaged after 10 minutes. This is true of video lecturers and face-to-face lecturers.

4. **Make it relevant.** As in a face-to-face class, the key to a successful flipped classroom is to relate problems to real-life applications. In Math 21, I had no problem creating applications for nursing students.

5. **Make it a two-way street.** Although I didnt try this, what Im thinking about for the next time I teach the class is to have students make their own videos or to come up with exam problems or critique other students work. This way, students have more control over their own learning.

6. **Have an introductory video about a flipped classroom.** I also didnt try this but to create a video where a flipped class is explained may help students understand how the class will run before the first day of the semester.

7. **Take attendance.** I didnt have a problem with attendance in my class but I can see it being a big problem for other classes. With a flipped class, the student must be present. I see a student missing a flipped class as being the same as missing a physics lab, which is very hard to make up. Also, give a grade reduction for students who do miss class.

8. **Unprepared students.** Students who dont watch videos or read the text before coming to class can cause a huge disruption. Here, make a seating chart and keep track of the students who are not prepared for the week. Offer one chance to redeem themselves and ask students questions the next class period instead.

9. **Dont assign too much homework.** Before starting the class, think about which problems test the concepts more than others. Remember, students are doing twice as much work outside of class while in a flipped class. You want to have clear goal on what you want

students to be able to do at the end of the semester. In short, developing learning goals is vital.

10. **Access to materials online.** Although it is 2015, some students still dont have regular access to computers. At the start of the semester, stress to students that they must have access to a computer either through family/friends or at public libraries or at school. Not having computer access is simply a no-go for a flipped class.

11. **Lots of prep work for the instructor.** When first faced with creating a hybrid class, you as the instructor need to devote a lot of time to preparing the class. The ideal time to prepare for a flipped class would be over the summer as the three months or so will give plenty of time to get videos, activities, announcements, syllabus, assessments, and homework ready to go.

12. **Monitor students work online.** With MML, it is very easy to see who is doing the work and who isnt. Stress to students deadlines and have a clear policy in place about late work. Also, talk to your IT specialists to see how you can monitor who watches video lecturers to make sure they are being watched.

13. **Large classes.** Although I have not had a personal experience in teaching a class larger than 50 students, I dont see how a flipped class of say, 300 students, would work. Sometimes, a class shouldnt be flipped.

14. **Allow feedback.** When teaching the course for the first time, I gave three feedback forms to my students to complete on ANGEL. The more feedback I received, the better the class went.

15. **Talk to fellow colleagues about their experiences.** Often times, the person next door to your office can provide you with a great wealth of experience. Dont be afraid to ask for advice.

16. **You are still the professor.** Even though the learning of the concepts happens outside the classroom, you are still the professor guiding students along and offering your wealth of knowledge in the face-to-face time with the students. Dont be discouraged about the new way of running your hybrid class vs. your traditional class, embrace it!

17. **If things go wrong.** Not everything will go right. Learn from your mistakes and try again.

Part 6. Conclusion

In conclusion, I can honestly say I have learned a great deal about myself through teaching my first hybrid class. When I made my first video and I heard it play back, I deleted it right away, as I couldnt believe how boring I sounded. It made me realize how I needed to change to get the students involved outside of class. The energy and enthusiasm I always have in a face-to-face class needed to be transferred in a new arena where students could think without my physical presence. I would highly encourage anyone who is thinking of trying the flipped classroom approach to not waste time thinking about, but to do it. It will make you stop and think about how to deliver your teaching in a new light.

Appendix A: Outline of Math 21

Week	Topics	Group Application Problems
Thursday, January 15, 2015 **Week I**	1. Set-up MML accounts. 2. Scientific Notation. 3. Linear Inequalities. 4. Compound Inequalities. 5. Absolute Value Equations. 6. Absolute Value Inequalities.	1. Unit Conversion Lab. 2. Bishop's Score. 3. Radon Gas Risks. 4. Growth Charts, Boys 2 - 20 years & Girls birth – 36 months. 5. Metric Mania. 6. Dosage Calculation Worksheet
Thursday, January 22, 2015 **Week II**	1. *MML Homework, § 2.4 – 2.7 Due.* 2. *Post-Test on Chapter 2 Due.* 3. Graphing Equations. 4. Introduction to Functions. 5. Graphing Linear Functions. 6. The Slope of a Line. 7. Equations of Lines. 8. Graphing Piece-wise-Defined Functions, Shifting, & Reflecting Graphs of Functions. 9. Graphing Linear Inequalities.	1. Fahrenheit vs. Celsius. 2. Incidence of Diabetes in Adults Ages 45 – 64. 3. Life Expectancy parts I-II. 4. Protein Intake. 5. Training Heart Rate Zones. 6. Graph of Heart Rate over Time.
Thursday, January 29, 2015 **Week III**	1. *MML Homework, § 3.1 – 3.7 Due.* 2. *Post-Test on Chapter 3 Due.* 3. Exponents & more on Scientific Notation. 4. Polynomials and Polynomial Functions. 5. Multiplying Polynomials. 6. The Greatest Common Factor and Factoring by Grouping. 7. Factoring Trinomials. 8. Factoring by Special Products. 9. Solving Equations by Factoring and Problem Solving.	Class will be working on MML homework, applications within MML.
Thursday, February 5, 2015 **Week IV**	1. *MML Homework, § 5.1 – 5.8 Due.* 2. *Post-Test on Chapter 5 Due.* 3. Rational Functions, Multiplying, & Dividing Rational Expressions. 4. Adding & Subtracting Rational Expressions. 5. Simplifying Complex Fractions. 6. Solving Equations Containing Rational Expressions. 7. Rational Equations & Problem Solving. 8. Variation & Problem Solving. 9. Radicals and Radical Functions. 10. Rational Exponents. 11. Simplifying Radical Expressions. 12. Adding, Subtracting, & Multiplying Radical Expressions. 13. Rationalizing Denominators &	1. Textbook applications. 2. Variation applications (non-nursing examples).

	Numerators of Radical Expressions. 14. Radical Equations & Problem Solving.	
Thursday, February 12, 2015 **Week V**	1. ***MML Homework, § 6.1 – 6.3, 6.5 – 6.7, & 7.1 – 7.6 Due.*** 2. ***Post-Test on Chapters 6 & 7 Due.*** 3. Solving Quadratic Equations by Completing the Square. 4. Solving Quadratic Equations by the Quadratic Formula. 5. Solving Equations by Using Quadratic Methods. 6. Quadratic Functions & Their Graphs. 7. Further Graphing of Quadratic Functions.	1. Additional Life Expectancy. 2. Heart Rate Data. 3. More on Heart Rate. 4. More on BMI. 5. Applications to Quadratics (non-Nursing examples).
Thursday, February 19, 2015 **Week VI**	1. ***MML Homework, § 8.1 – 8.3 & 8.5 – 8.6 Due.*** 2. ***Post-Test on Chapter 8 Due.*** 3. Exponential Functions. 4. Exponential Growth & Decay. 5. Logarithmic Functions. 6. Properties of Logarithms. 7. Common Logarithms, Natural Logarithms, & Change of Base Formula. 8. Exponential & Logarithmic Equations & Problem Solving.	1. Calculating pH. 2. Spread of a Virus. 3. Compound Interest applications (non-nursing examples). 4. Scientific Applications with continuous compound formula. 5. Cancer Cell Simulation.
Thursday, February 26, 2015 **Week VII**	1. ***MML Homework, § 9.3 – 9.8 Due.*** 2. ***Post-Test on Chapter 9 Due.*** 3. Sequences. 4. Series. 5. The Binomial Theorem. 6. Review for Final Exam. 7. ***Post-Test on Chapter 11 Due on 3/1 at 11:59 pm.***	1. Eliminating Medicine from the Bloodstream.

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On the zero divisor graphs of Galois rings

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Abstract

Let R be a Galois ring. The subset of zero divisors of R is studied with specific emphasis on the graph theoretical properties. The zero divisor graphs determined by equivalence classes of the zero divisors of the ring are also explored.

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Abstract. Let R be a Galois ring. The subset of zero divisors of R is studied with specific emphasis on the graph theoretical properties. The zero divisor graphs determined by equivalence classes of the zero divisors of the ring are also explored.

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1. Introduction

In this section, we summarize some well known results on the zero divisor graphs of commutative rings. The definition of terms and standard notations can be obtained from [1, 4, 8]. The concept of a zero divisor graph was proposed by Ivstan Beck in [3]. According to him, $G(R) = R$ such that the distinct vertices u and v are adjacent iff $uv = 0$. Since the zero element is adjacent to every other vertex, the graph is connected with $\mathrm{diam}(G(R)) \leq 2$. He conjectured that the chromatic number coincides with the clique number of $G(R)$. Later, D.D. Anderson and M. Naseer in [1] provided an example where the clique number is strictly less than the chromatic number. D.F. Anderson and P.S. Livingston [2] simplified Beck's graph. They considered the non zero zero divisors as the vertices of the graph $\Gamma(R)$ and the adjacency concept is similar to Beck's. Amongst their major findings, the graph of commutative ring R is connected with diam $(\Gamma(R)) \leq 3$. S.B. Mulay in [6] introduced the zero divisor graph $\Gamma_E(R)$ of equivalent vertices. The graph $\Gamma_E(R)$ is connected with $\mathrm{diam}(\Gamma_E(R)) \leq 3$. This graph can be finite even if R is infinite, thus it is less "noisy" than its predecessors. S.P. Redmond in [7], originated a zero divisor graph with respect to an ideal I. The distinct vertices u and v are adjacent with $uv \in I$. In this paper, some graph theoretical properties of Galois rings have been determined.

2. Zero divisor graphs of Galois rings

A ring R is said to be Galois, if its subset of all the zero divisors (including zero) forms a principal ideal. It is an extension of the ring of integers modulo p^k (where p is a prime integer and k is a positive integer). The extension is usually represented by $\mathbb{Z}_{p^k}[x]/ < f(x) >$ where $f(x) \in \mathbb{Z}_{p^k}[x]$ is a monic polynomial of degree r and irreducible over $\mathbb{Z}_p$. It is denoted by $GR(p^{kr}, p^k)$. A Galois ring can be trivial depending on whether k or r equals to 1, otherwise it is nontrivial.

Let R_o be a Galois ring and $Z(R_o)$ be its subset of zero divisors (including zero). Then $Z(R_o)$

is a unique maximal ideal of R_o and is therefore the Jacobson radical of R_o. A graph $\Gamma(R_o)$ is associated to R_o with vertices $Z(R_o)^* = Z(R_o) \setminus \{0\}$, the set of nonzero zero divisors of R_o. The graph theoretical properties are used to illuminate the structure of $Z(R_o)$. Two distinct vertices $u_1, u_2 \in Z(R_o)^*$ are adjacent if $u_1 u_2 = 0$ and equivalent if $\text{Ann}_{R_o}(u_1) = \text{Ann}_{R_o}(u_2)$. The graph of equivalent vertices in R_o, $\Gamma_E(R_o)$ is simple with vertex set $Z(R_o)^*/\sim$ such that $[u_1], [u_2] \in Z(R_o)^*/\sim$, adjacent provided $u_1 u_2 = 0$.

Let $u \in Z(R_o)^*$ and $s \neq 1$ be a unit in R_o. In $\Gamma(R_o)$, the vertices u and su are distinct and $[u] = [su]$ in $\Gamma_E(R_o)$. So $\Gamma_E(R')$ is less noisy than $\Gamma(R)$, (see[5]).

In this paper, we investigate the connectedness of $\Gamma(R_o)$, $\Gamma_E(R_o)$ and determine the diameter, girth and binding number of $\Gamma(R_o)$ and $\Gamma_E(R_o)$.

Theorem 2.1. *(See e.g [2]) Let R be a commutative ring (not necessarily Galois). Then $\Gamma(R)$ is finite iff R is finite or an integral domain.*

Remark 2.2. *From the above theorem, we notice that $\Gamma(R_o)$ is finite because R_o is finite.*

Remark 2.3. *If $R_o = GR(p^r, p)$, then $Z(R_o)^*$ is empty. It will therefore be of interest to characterize the zero divisor graphs of $R_o = GR(p^{kr}, p^k)$, where $k > 1, r \geq 1$*

Remark 2.4. *If p is prime, then $\Gamma(GR(p^2, p^2))$ is a complete graph on $p - 1$ vertices and $\Gamma_E(GR(p^2, p^2))$ is a single vertex. But $\Gamma(GR(p^3, p^3))$ is messy while $\Gamma_E(GR(p^3, p^3))$ is a single edge. Moreover, $\Gamma(GR(p^4, p^4))$ is even messier while $\Gamma_E(GR(p^4, p^4))$ is a star graph.*

Proposition 2.5. *Let $R_o = GR(p^k, p^k)$. Suppose α is a unit in R_o and $|\Gamma(R_o)| = p^l \alpha$, then the degree of $p^l \alpha$,*

$$deg(p^l \alpha) = \begin{cases} p^l - 1 \text{ if } 2l < k \\ \\ p^l - 2 \text{ if } 2l \geq k \end{cases}$$

Proof. Given that α is a unit in R_o, then $(\alpha, p^k) = 1$. So, all the vertices adjacent to $p^l \alpha$ in the set $pGR(p^k, p^k)$ of the graph $\Gamma(R_o)$ are the same vertices adjacent to p^l. Now, to find the degree of $p^l \alpha$, it suffices to find all the vertices adjacent to p^l. Let n be the number of vertices adjacent to p^l in $\Gamma(R_o)$. The first term in this sequence of vertices is p^{k-l} and the nth term is $p^{k-l} + (n-1)p^{k-l}$. Since the last term in the sequence is $p^k - p^{k-l}$, it easily follows that $p^{k-l} + (n-1)p^{k-l} = p^k - p^{k-l}$ leading to $n = p^l - 1$ if $p^{k-l} > p^l$ or $k > 2l$. If $p^{k-l} \leq p^l$ or $k \leq 2l$, then $p^{2l-k}(p^{k-l}) = p^l$ is adjacent to itself, so that $\deg(p^l) = p^l - 2$. ∎

Proposition 2.6. *Given that $R_o = GR(p^{kr}, p^k)$ and α is a unit in R_o, then the degree of $p^{lr} \alpha$,*

$$deg(p^{lr} \alpha) = \begin{cases} p^{lr} - 1 \text{ if } 2l < k \\ \\ p^{lr} - 2 \text{ if } 2l \geq k \end{cases}$$

Proposition 2.7. *Let $R_o = GR(p^k, p^k)$, $k \geq 3$. Then,*

$$\Gamma(R_o) = \begin{cases} (p^{\frac{k}{2}} - 1) - partite \text{ if } k \text{ is even} \\ \\ p^{\frac{k-1}{2}} - partite \text{ if } k \text{ is odd} \end{cases}$$

Proof. Obviously $Z(R_o)^* = Z(R_o) - \{0\} = pR_o - \{0\}$.

Case (i): k is an even integer. We partition $Z(R_o)^*$ into the following subsets:

$$V_1 = Z(R_o)^* - \{j(p^{\frac{k}{2}}), 2 \leq j \leq p^{\frac{k}{2}} - 1\};$$

$$V_j = \{j(p^{\frac{k}{2}})\}, 2 \leq j \leq p^{\frac{k}{2}} - 1.$$

For $1 \leq i \leq p^{\frac{k}{2}} - 1$, $V_i \neq \emptyset$, V_1 does not contain adjacent vertices, each V_j is singleton,

$$V_1 \cap V_j = \emptyset, 2 \leq j \leq p^{\frac{k}{2}} - 1;$$

$$V_j \cap V_l = \emptyset, j \neq l, 2 \leq j, l \leq p^{\frac{k}{2}} - 1.$$

Finally

$$Z(R_o)^* = V_1 \cup (\cup_{j=2}^{p^{\frac{k}{2}}-1} V_j) = \cup_{i=1}^{p^{\frac{k}{2}}-1} V_i.$$

Thus $\Gamma(R_o)$ is $p^{\frac{k}{2}} - 1$ partite.

Case (ii): k is an odd integer. We partition $Z(R_o)^*$ into the following subsets:

$$V_1 = Z(R_o)^* - \{(j-1)(p^{\frac{k+1}{2}}), 2 \leq j \leq p^{\frac{k-1}{2}}\};$$

$$V_j = \{(j-1)(p^{\frac{k+1}{2}})\}, 2 \leq j \leq p^{\frac{k-1}{2}}$$

. For $1 \leq i \leq p^{\frac{k-1}{2}}$, $V_i \neq \emptyset$, V_1 does not contain adjacent vertices, each V_j is singleton,

$$V_1 \cap V_j = \phi, 2 \leq j \leq p^{\frac{k-1}{2}};$$

$$V_j \cap V_l = \emptyset, j \neq l, 2 \leq j, l \leq p^{\frac{k-1}{2}}.$$

Finally

$$Z(R_o)^* = V_1 \cup (\cup_{j=2}^{p^{\frac{k-1}{2}}} V_j) = \cup_{i=1}^{p^{\frac{k-1}{2}}} V_i.$$

Thus $\Gamma(R_o)$ is $p^{\frac{k-1}{2}}$ partite

$\blacksquare$

Example 1.

Let $R_o = GR(8,8)$. Then $Z(R_o)^* = \{2,4,6\}$. $V_1 = \{2,6\}, V_2 = \{4\}$. So $\Gamma(R_o)$ is bipartite.

Example 2

Let $R_o = GR(81,81)$. Then $Z(R_o)^* = \{3,6,9,12,15,18,21,24,27,30,$
$36,39,42,45,48,51,54,57,60,63,66,69,72,75,78\}$. $V_1 = \{3,6,9,12,15,21,24,30,$
$33,39,42,48,51,57,60,66,69,75,78\}$, $V_2 = \{18\}, V_3 = \{27\}, V_4 = \{36\}, V_5 = \{45\}, V_6 = \{54\},$
$V_7 = \{63\}, V_8 = \{72\}$. So $\Gamma(R_o)$ is octapartite.

Proposition 2.8. *Suppose $R_o = GR(p^k, p^k), k \geq 4$. Then*
i)The diameter, $diam(\Gamma(R_o)) = 2$
ii) The girth, $gr(\Gamma(R_o)) = 3$
iii)The binding number,

$$b(\Gamma(R_o)) = \begin{cases} \frac{p^{\frac{k}{2}}-2}{p^{k-1}-p^{\frac{k}{2}}+1} & \text{if } k \text{ is even} \\ \\ \frac{p^{\frac{k-1}{2}}-1}{p^{k-1}-p^{\frac{k-1}{2}}} & \text{if } k \text{ is odd} \end{cases}$$

Proof. (i) and (ii) are clear. To prove (iii), we consider the two cases separately. If k is even, we consider

$$V_1 = Z(R_o)^* - \{j(p^{\frac{k}{2}}), 2 \leq j \leq p^{\frac{k}{2}} - 1\}$$

and

$$N(V_1) = \{j(p^{\frac{k}{2}}), 2 \leq j \leq p^{\frac{k}{2}} - 1\}.$$

Notice that

$$\mid V_1 \mid = p^{k-1} - p^{\frac{k}{2}} + 1$$

and

$$\mid N(V_1) \mid = p^{\frac{k}{2}} - 2$$

, so that

$$b(\Gamma(R_o)) = \frac{\mid N(V_1) \mid}{\mid V_1 \mid} = \frac{p^{k/2} - 2}{p^{k-1} - p^{\frac{k}{2}} + 1}.$$

If k is odd, we consider

$$V_1 = Z(R_O)^* - \{(j-1)(p^{\frac{k+1}{2}}), 2 \le j \le p^{\frac{k-1}{2}}\}$$

and

$$N(V_1) = \{(j-1)(p^{\frac{k+1}{2}}), 2 \le j \le p^{\frac{k-1}{2}}\}.$$

Notice that

$$\mid V_1 \mid = p^{k-1} - p^{\frac{k-1}{2}}$$

and

$$\mid N(V_1) \mid = p^{\frac{k-1}{2}-1}$$

, so that

$$b(\Gamma(R_o)) = \frac{\mid N(V_1) \mid}{\mid V_1 \mid} = \frac{p^{\frac{k-1}{2}} - 1}{p^{k-1} - p^{\frac{k-1}{2}}}.$$

∎

Remark 2.9. *Easy computations yield $\lim_{k \to \infty} b(\Gamma(R_o)) = 0$. This implies that as k increases (whether odd or even), more vertices become non adjacent.*

Lemma 2.10. *Let $R_o = GR(p^r, p)$, then $Z(R_o)^* = \emptyset$*

Proof. Easy

∎

Proposition 2.11. *Let $R_o = GR(p^{2r}, p^2)$. Then $\Gamma(R_o) = K_{p^r - 1}$ and $\Gamma_E(R_o)$ is a single vertex.*

Proof. Since $((Z(R_o))^2 = 0$, each zero divisor is adjacent to the other. But

$$\mid Z(R_o)^* \mid = p^r - 1$$

, so that $\Gamma(R_o)$ is complete on $p^r - 1$ vertices. That $\Gamma_E(R_o)$ is a single vertex follows from the fact that $\text{Ann}(Z(R_o)) = Z(R_o)$

∎

Remark 2.12. *Let $R_o = GR(p^{3r}, p^3)$. Then $\Gamma(R_o)$ is noisy while $\Gamma_E(R_o)$ is a single edge.*

Proposition 2.13. *Given that $R_o = GR(p^{kr}, p^k), k \ge 4$. Then the clique number,*

$$\omega(\Gamma_E(R_o)) = \begin{cases} \frac{k}{2} \text{ if } k \text{ is even} \\ \\ \frac{k+1}{2} \text{ if } k \text{ is odd} \end{cases}$$

Proof. It suffices to find a maximal complete subgraph of $\Gamma_E(R_o)$. Let s be a unit in R_o.

Case I: k is even.
We show that $\Gamma_E(R_o)$ has a maximal complete subgraph S with vertices

$$\{[p^\iota s] = [p^\iota], \frac{k}{2} \le \iota \le k - 1\}.$$

Suppose on the contrary that S is not maximal in $\Gamma_E(R_o)$ and that there exists a maximal

complete subgraph $S' \subset \Gamma_E(R_o)$ so that $S \subset S'$. Without loss of generality, assume that $p^i \in S'$ where

$$
\begin{cases}
0 < i < \frac{k}{2}, \ k \text{ is even} \\[2mm]
0 < i < \frac{k-1}{2}, \ k \text{ is odd}
\end{cases}
$$

then there exists some $j > i > 0$ so that

$$
p^i \cdot p^{k-1-j} = p^{k-1+i-j}
$$
$$
\neq 0.
$$

So S' is not complete, a contradiction.

Case II: k is odd.
Using similar steps as Case I above, we can show that $\Gamma_E(R_o)$ contains a maximal complete subgraph with vertices

$$
\{[p^\iota s] = [p^\iota], \frac{k-1}{2} \leq \iota \leq k-1\}.
$$

∎

Proposition 2.14. *Let* $R_o = GR(p^{kr}, p^k), k \geq 4$. *Then*
i) the diameter,diam $(\Gamma_E(R_o)) = 2$
ii) the girth, $gr(\Gamma_E(R_o)) = 3$
iii)the binding number,

$$
b(\Gamma_E(R_o)) =
\begin{cases}
\frac{k-4}{k} \ \text{if } k \text{ is even} \\[2mm]
\frac{k-5}{k-1} \ \text{if } k \text{ is odd}
\end{cases}
$$

Proof. (i) and (ii) are easy. To prove (iii), we consider the two cases separately. When k is even, then from Proposition 2 ,

$$
\mid V_1 \mid = \frac{k}{2}
$$

while

$$
\mid N(V_1) \mid = \frac{k-4}{2},
$$

so that

$$
b(\Gamma(R_o)) = \frac{k-4}{k}.
$$

Next, when k is odd,

$$
\mid V_1 \mid = \frac{k-1}{2}
$$

while

$$
\mid N(V_1) \mid = \frac{k-5}{2}.
$$

The result then easily follows.

∎

Corollary 2.15. *Suppose* $R_o = GR(p^{kr}, p^k)$ *where* $k \geq 4$. *Then*

$$
\Gamma_E(R_o) =
\begin{cases}
\frac{k}{2} \ \text{partite if } k \text{ is even} \\[2mm]
\frac{k+1}{2} \ \text{partite if } k \text{ is odd}
\end{cases}
$$

Proof. Case I: k is even.
$\Gamma_E(R_o)$ is partitioned into the following subsets.

$$V_1 = \{(Z(R_o))^\iota : 1 \le \iota \le \frac{k}{2}\}$$

$$V_i = \{(Z(R_o))^i\}, \frac{k}{2} < i \le k-1.$$

For each i, $V_1 \cap V_i = \emptyset$ and the V_i are mutually disjoint. Moreover,

$$V_1 \cup (\cup_{i=k/2}^{k-1} V_i) = \Gamma_E(R_o)$$

. The result then follows by counting the disjoint subsets of $\Gamma_E(R_o)$.

Case II: k is odd.
In this case, $\Gamma_E(R_o)$ is partitioned into the following subsets.

$$V_1 = \{(Z(R_o))^\iota : 1 \le \iota \le \frac{k-1}{2}\}$$

$$V_i = \{(Z(R_o))^i\}, \frac{k-1}{2} < i \le k-1.$$

The rest of the proof involves steps similar to Case I with some slight modifications.

$\blacksquare$

Proposition 2.16. *Let $R_o = GR(p^{kr}, p^k), k \ge 3$. Then*
i) the diameter,diam $(\Gamma(R_o)) = 2$
ii) the girth, $gr(\Gamma(R_o)) = 3$
iii) the binding number,

$$b(\Gamma(R_o)) = \begin{cases} \frac{p^{(\frac{k}{2})r}-2}{p^{(k-1)r}-p^{(\frac{k}{2})r}+1} & \text{if } k \text{ is even} \\[2em] \frac{p^{(\frac{k-1}{2})r}-1}{p^{(k-1)r}-p^{(\frac{k-1}{2})r}} & \text{if } k \text{ is odd} \end{cases}$$

Proof. (i) and (ii) are elementary. To prove (iii), begin with the case when k is even. Let $\epsilon_1, ..., \epsilon_r \in R_o$ with $\epsilon_1 = 1$ such that

$$\overline{\epsilon_1}, ..., \overline{\epsilon_r} \in R_o/pR_o$$

form a basis for R_o/pR_o regarded as a vector space over its prime subfield F_p. By the definition of V_1 in the previous Proposition,

$$N(V_1) = V_{\sum a_i \epsilon_i}.$$

So

$$\mid N(V_1) \mid = p^{(\frac{k}{2})r} - 2.$$

Also

$$\begin{aligned} \mid V_1 \mid &= \mid Z(R_o)^* \mid \\ &= \mid V_{\sum a_i \epsilon_i} \mid \\ &= p^{(k-1)r} - 1 - (p^{(k-1)r} - 2) \\ &= p^{(k-1)r} - p^{\frac{k}{2}r} + 1. \end{aligned}$$

Let k be odd. Then

$$\begin{aligned} \mid N(V_1) \mid &= \mid V_{\sum a_i \epsilon_i} \mid \\ &= p^{\frac{k-1}{2}r} - 1. \end{aligned}$$

Also

$$\begin{aligned}
\mid V_1 \mid &= \mid Z(R_o)^* - X \mid \\
&= \mid Z(R_o)^* \mid - \mid X \mid \\
&= p^{(k-1)r} - 1 - (p^{(\frac{k-1}{2})r} - 1) \\
&= p^{(k-1)r} - p^{(\frac{k-1}{2})r}
\end{aligned}$$

∎

Proposition 2.17. *Let $R_o = GR(p^{kr}, p^k)$, where $k \geq 3$ and $r \in \mathbb{Z}^+$. Then*

$$\Gamma(R_o) = \begin{cases} p^{(\frac{k}{2})r} - 1 \text{ partite if } k \text{ is even} \\ \\ p^{(\frac{k-1}{2})r} \text{ partite if } k \text{ is odd} \end{cases}$$

Proof. Clearly

$$Z(R_o)^* = Z(R_o) \setminus \{0\} = pR_o \setminus \{0\}.$$

Let $\epsilon_1, ..., \epsilon_r \in R_o$ with $\epsilon_1 = 1$ such that

$$\overline{\epsilon_1}, ..., \overline{\epsilon_r} \in R_o/pR_o$$

form a basis for R_o/pR_o regarded as a vector space over its prime subfield F_p. We consider the two cases separately.

Case I:k is an even integer.
We partition $Z(R_o)^*$ into the following subsets.

$$X = \{a_i \epsilon_i, 1 \leq i \leq r, a_i \in \{0, j(p^{\frac{k}{2}})\}, 1 \leq j \leq p^{\frac{k}{2}} - 1\}$$

$$V_{\sum a_i \epsilon_i} = X \setminus \{0, p^{\frac{k}{2}}\}$$

$$V_1 = Z(R_o)^* \setminus V_{\sum a_i \epsilon_i}\}$$

For each $i = 1, ..., r$, $V_{\sum a_i \epsilon_i} \neq \emptyset$. Each of the $V_{\sum a_i \epsilon_i} \neq \emptyset$ contains no adjacent vertices, $V_1 \cap V_{\sum a_i \epsilon_i} = \emptyset$. The sets $V_{\sum a_i \epsilon_i}$ are mutually disjoint. Moreover $Z(R_o)^* = V_1 \cup (\cup_r V_{\sum a_i \epsilon_i})$. Thus $\Gamma(R_o)$ is $p^{(\frac{k}{2})r}$ partite.

Case II:k is an odd integer.
We partition $Z(R_o)^*$ into the following subsets.

$$X = \{a_i \epsilon_i, 1 \leq i \leq r, a_i \in \{0, (j-1)(p^{\frac{k+1}{2}})\}, 1 \leq j \leq p^{\frac{k-1}{2}}\}$$

$$V_{\sum a_i \epsilon_i} = X \setminus \{0\}$$

$$V_1 = Z(R_o)^* \setminus X.$$

The rest of the proof is similar to the previous case with slight modifications.

∎

3. Automorphisms of zero divisor graphs of Galois rings

Consider the integer $k \geq 2$ and a positive integer r. Distinct ring automorphisms of $GR(p^{kr}, p^k)$ induce distinct graph automorphisms of $\Gamma(GR(p^{kr}, p^k))$, because $GR(p^{kr}, p^k)$ is a finite ring which is not a field, (see [2]). A graph automorphism, f of a graph $\Gamma(R_o)$ is a bijection $f : \Gamma \to \Gamma$ which preserves adjacency. The set $\mathrm{Aut}(\Gamma)$ of all graph automorphisms of Γ forms a group under the usual composition of functions, (see [2]). If $\mid \Gamma \mid = p^k$, then in the obvious way, $\mathrm{Aut}(\Gamma)$ is isomorphic to a subgroup of S_{p^k} and clearly $\mathrm{Aut}(K_{p^k}) \cong S_{p^k}$. Infact, for a graph Γ of order p^k, $\mathrm{Aut}(\Gamma) \cong S_{p^k}$ iff $\Gamma = K_{p^k}$. Now, by restricting each $f \in \mathrm{Aut}(R_o)$ to $Z(R_o)^*$ we obtain a natural group homomorphism $\phi : \mathrm{Aut}(R_o) \to \mathrm{Aut}(\Gamma(R_o))$.

Theorem 3.1. *(See [2]) Let $R_o = GR(p^{kr}, p^k)$ and let $f \in Aut(R_o)$. If $f(x) = x, \forall x \in Z(R_o)$, then $f = 1_{R_o}$. Thus, $\phi : Aut(R_o) \to Aut(\Gamma(R_o))$ is a monomorphism.*

Now, consider $R_o = GR(p^k, p^k)$. For $p^k \geq 4$, $k \neq 1$, let

$$X = \{j \in \mathbb{Z} \mid 1 < j < p^k, j \mid p^k\}.$$

For each, $j \in X$, let

$$V_j = \{l \in \mathbb{Z} \mid 1 < l < p^k, (l, p^k) = j\}.$$

Note that

$$Z(GR(p^k, p^k))^*$$

is the disjoint union of $V_j s$. Notice that two vertices have the same degrees iff they are in the same V_d (See [2]).

Proposition 3.2. *Consider the integer $k \geq 2$. Then*

$$\mid Aut(\Gamma(p^{kr}, p^k)) \mid = \begin{cases} \prod_{l=2}^{k}(2^{(k-l)r})! \text{ if } p = 2 \\ \\ \prod_{l=2}^{k}(p^{(k-l)r}(p^r - 1))! \text{ if } p \neq 2 \end{cases}$$

Proof. Case I: $r = 1$.
Let $p = 2$. Set $X = \{2, 4, ..., 2^{k-1}\}$. Then $V_2 = \{2t \mid t \text{ is odd}\}$, $V_4 = \{4t \mid t \text{ is odd}\}$,...,$V_{2^{k-1}} = \{2^{k-1}t \mid t \text{ is odd}\}$. Upon counting, $\mid V_2 \mid = 2^{k-2}$, $\mid V_4 \mid = 2^{k-3}$ and continuing in a similar manner, $\mid V_{2^{k-2}} \mid = 2$, and $\mid V_{2^{k-1}} \mid = 1$. So $\mathrm{Aut}(\Gamma(GR(2^k, 2^k))) \cong \prod_{l=2}^{k} S_{2^{k-l}}$ and the result easily follows.
Let $p \neq 2$. Set $X = \{p, p^2, ..., p^{k-1}\}$. Then $V_p = \{pt \mid (t, p) = 1\}$, $V_{p^2} = \{p^2t \mid (t, p^2) = 1\}$,...,$V_{p^{k-1}} = \{p^{k-1}t \mid (t, p^{k-1}) = 1\}$. Upon counting, $\mid V_p \mid = p^{k-2}(p - 1)$, $\mid V_{p^2} \mid = p^{k-3}(p - 1)$ and continuing in a similar manner, $\mid V_{p^{k-1}} \mid = p - 1$. So $\mathrm{Aut}(\Gamma(GR(p^k, p^k))) \cong \prod_{l=2}^{k} S_{p^{k-l}(p-1)}$ and the result follows immediately.
Case II: $r > 1$.
Let $\epsilon_1, ..., \epsilon_r \in R_o$ with $\epsilon_1 = 1$ such that $\overline{\epsilon_1}, ..., \overline{\epsilon_r} \in R_o/Z(R_o)$ form a basis for $R_o/Z(R_o)$ regarded as a vector space over its prime subfield $GF(p)$. For $p = 2$, let $X = \{2, 4, ..., 2^{k-1}\}$ and $V_{\sum_{a_i \epsilon_i}}$ where $a_i \in X$ be the disjoint vertices, then clearly $\mathrm{Aut}(\Gamma(GR(2^{kr}, 2^k))) = S_{2^{(k-l)r}}$. The steps are similar for the case when p is odd.   ∎

4. A class of finite rings

Let R_o be the Galois ring of the form $GR(p^{kr}, p^k)$. For each $i = 1, ..., h$, let $u_i \in Z(R_o)$ such that U is an h dimensional R_o- module generated by $u_1, ..., u_h$ so that $R = R_o \oplus U$ is an additive group. On this group, define multiplication by the following relations:

: (i)If $k = 1, 2$ then $pu_i = u_i u_j = u_j u_i = 0$, $u_i r_o = r_o u_i$.

: (ii)If $k > 3$ then $p^{k-1}u_i = 0$, $u_iu_j = p^2\gamma_{ij}$, $u_i{}^k = u_i{}^{k-1}u_j = u_iu_j{}^{k-1} = 0$, $u_ir_o = r_ou_i$.
where $r_o, \gamma_{ij} \in R_o$, $1 \le i, j \le h$, p is a prime integer, n and r are positive integers. Moreover if $u_i|_U$, then the additive order of u_i is p.

It can be shown that the multiplication turns the additive group into a commutative ring with identity. The structure of the Von Neumann regular elements of the ring is well known. We present some results on the structure its zero divisors.

Proposition 4.1. *Let R be a ring constructed in this section. If $Z(R) = pR_o \oplus U$; $ann(Z(R_o)) = p^{n-1}R_o \oplus U$; $J^{n-1} = p^{n-1}R_o$.*
If
i) $x \in ann(J)$ then $deg(x) = \mid J \mid -2$
ii)$y \in Z(R)$ but $y \in R - ann(Z(R))$, then $deg(y) = \mid ann(Z(R)) \mid -1$

Proposition 4.2. *Let R be a ring constructed in this section. Then*
i)

$$\Gamma(R) = \begin{cases} p^{(\frac{k}{2}+h)r} - 1 \text{ partite if } k \text{ is even} \\\\ p^{(\frac{k-1+2h}{2})r} \text{ partite if } k \text{ is odd} \end{cases}$$

ii)$diam(\Gamma(R)) = 2$
iii) $gr(\Gamma(R)) = 3$
iv)

$$b(\Gamma(R)) = \begin{cases} \dfrac{p^{(\frac{k}{2}+h)r}-2}{p^{(k-1+h)r}-p^{(\frac{k}{2}+h)r}} \text{ if } k \text{ is even} \\\\ \dfrac{p^{(\frac{k-1}{2}+h)r}-1}{p^{(k-1+h)r}-p^{(\frac{k-1}{2}+h)r}} \text{ if } k \text{ is odd} \end{cases}$$

5. Conclusion

This study reveals, that it is possible to generalize the graph theoretical properties of Galois rings. The zero divisor graphs of Galois rings are symmetrical as confirmed from their automorphisms. It would be interesting to investigate, whether these properties extend to the zero divisor graphs of the idealizations of the Galois rings.

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IMHOTEP

Renewable energy biodiesel: A mathematical approach from ecology to production

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Abstract

Biodiesel is one of promising renewable energy source and used as an alternative of conventional hydrocarbon fuels. *Jatropha curcas* plant oil (JCPO) is the most cost effective sources of biodiesel. The plant can be cultivated in wastelands and grows on almost any type of territory, even on sandy and saline soils. Judicious agricultural practices and effective crop management of *Jatropha curcas* is preliminary requisite to get maximum yield of oil. Production of biodiesel by transesterification of Jatropha oil significantly depends on four reaction parameters viz., reaction time, temperature, oil to alcohol molar ratio and stirrer speed. In this work, we have formulated a mathematical model of *Jatropha curcas* plant, which is affected by many type of pest with the aim to control the pest through *Nuclear Polyhedrosis Virus* (NPV). Here we have also concentrated on insecticide spraying as controlling measure to reduce the pest, to get maximum yield of Jatropha seeds, which gives Jatropha oil. We have also shown the effect of different variants on mass transfer in biodiesel production from JC oil and how the control theoretic approach flags the maximum production of biodiesel under the mathematical paradigm. Our analytical results provide an idea of the cost effective faster rate of biodiesel production, which satisfies our numerical conclusions.

Renewable energy biodiesel: A mathematical approach from ecology to production

Priti Kumar Roy, Jahangir Chowdhury and Fahad Al Basir

Abstract. Biodiesel is one of promising renewable energy source and used as an alternative of conventional hydrocarbon fuels. *Jatropha curcas* plant oil (JCPO) is the most cost effective sources of biodiesel. The plant can be cultivated in wastelands and grows on almost any type of territory, even on sandy and saline soils. Judicious agricultural practices and effective crop management of *Jatropha curcas* is preliminary requisite to get maximum yield of oil. Production of biodiesel by transesterification of Jatropha oil significantly depends on four reaction parameters viz., reaction time, temperature, oil to alcohol molar ratio and stirrer speed. In this work, we have formulated a mathematical model of *Jatropha curcas* plant, which is affected by many type of pest with the aim to control the pest through *Nuclear Polyhedrosis Virus* (NPV). Here we have also concentrated on insecticide spraying as controlling measure to reduce the pest, to get maximum yield of Jatropha seeds, which gives Jatropha oil. We have also shown the effect of different variants on mass transfer in biodiesel production from JC oil and how the control theoretic approach flags the maximum production of biodiesel under the mathematical paradigm. Our analytical results provide an idea of the cost effective faster rate of biodiesel production, which satisfies our numerical conclusions.

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Keywords. *Jatropha curcas* oil, Transesterification, Biodiesel, Reaction parameter, Mathematical modelling, Optimal control.

1. Introduction

In today's world, petroleum is clearly the most important energy source, providing more than half of the world's power, as well as being a basic material used in the manufacture of fertilizer, synthetic fibers, plastics and synthetic rubber. Demand is ever increasing worldwide, yet petroleum resources are finite and non-renewable. Concerns about diminishing supplies, rising cost and environmental problems of hydrocarbon fuels have been motivating researchers globally to seek more extensively alternative, renewable energy sources to harness future energy demands. Jatropha oil is environmentally safe, cost effective renewable source of non-conventional energy and a promising substitute for diesel. Biodiesel can be derived from Jatropha oil and it is clean-burning, biodegradable, natural and it can help in reducing mineral oil dependence also. It is one of the promising and profitable agro forestry crop grown for its bio-diesel production

[1, 2]. The cultivation of Jatropha plant has been considered as one of the most promising solutions to the problems created by climate change and energy insecurity in developing countries for several years. Also, Jatropha is a multipurpose shrub of significant economic importance because of its several potential industrial and medicinal uses. Jatropha is depicted as miracle tree that could simultaneously produce biodiesel, reclaim waste land and enhance rural development without compromising food production or ecosystem services [3, 4, 5].

Jatropha curcas seed oil is proven to be toxic to many microorganisms, insects and animals. Despite its toxicity, Jatropha plant is not pest and disease resistant. The major pests and diseases affecting Jatropha are: 1) the leaf miner *Stomphastis thraustica*, 2) the leaf and stem miner *Pempelia morosalis*, and 3) the shield-backed bug *Calidea panaethiopica*, which can cause flower and fruit abortion. Damage from these pests particularly during the second year after the plantations and before later receding is dangerous [6].

To get sufficient oil from this plant, protection from different diseases is essential. In this respect study of disease dynamics of Jatropha plant can be done with the help of an eco-epidemiological mathematical model for controlling and/or minimizing diseases. There are very few information available on the eco-epidemiological mathematical model for the study of pest and disease dynamics to save *Jatropha curcas* plant from different pests. Mathematical models for pest control have been proposed and studied by several researchers [7, 8]. They have studied the effect of environmental fluctuations on a sterile insect release method. Bhattacharya and Bhattacharya [9] have designed and analysed a mathematical model under sterile insect population and pesticide. Pest management models through control strategies have been designed and analysed by many researchers mathematically oftentimes[10, 11, 12].

Vegetable oil from Jatropha plant is the most viable source for the production of biodiesel. It is significant to point out that the biodiesel from Jatropha plant has the requisite potential of providing a promising and commercially viable alternative to diesel oil since it has desirable physicochemical and performance characteristics comparable to diesel. It is utmost important to protect the crop from pest in order to obtain uninterupted supply of Jatropha fruits which is the key raw material for Jatropha seeds.

Oil which is extracted from the seeds of *Jatropha curcas* plant is converted to biodiesel by transesterification reaction with the help of chemical or biological catalyst. This transesterification reaction depends on some reaction conditions such as temperature, catalyst concentration, speed of stirrer, molar ratios of oil and alcohol. To get maximum conversion, certain major physical and chemical property of the reaction system should be maintained properly [13], [14], [15]. Mathematical modeling can be helpful and its implementation in reality would be beneficial in this regard. It was found that in the production of biodiesel from Jatropha oil by transesterification, the high FFA in the Jatropha oil can react with the catalyst (KOH) to produce soap [4], [16]. This reaction causes lower yield of biodiesel, washing difficulties and purity of biodiesel.

Thus, the main objectives of this article are two fold. Initially we formulate an Eco-epidemiological mathematical model for studying the *Jatropha curcas* plant pest management for controlling and/or minimizing damages of the crop and successively to formulate a mathematical model to optimize the production of biodiesel from Jatropha oil with chemical catalysts by controlling reaction parameters. We determine the optimal control condition with the help of simple mathematical technique and discuss the relevant analysis with respect to the specified time interval, by varying control factors to maximize the biodiesel production.

2. Formulation of The Mathematical Model for Jatropha Pest Control

The model, we analyze in this paper, has four populations, viz.

(i) The plant population, J,

(ii) The susceptible pest population, S,

(iii) The infected pest, I and

(iv) The virus, V.

The following assumptions are taken to formulate the mathematical model:

(A1) In the absence of the pest, plant population grows according to a logistic curve with carrying capacity $K_J(K_J > 0)$ and with an intrinsic growth rate constant R_J. Thus the rate equation for Jatropha plant is:

$$\frac{dJ}{dT} \;=\; R_J J(1 - \frac{J}{K_J}).$$

(1)

Now, Jatropha plant gets infected with pest thereby causing considerable crop reduction. Let λ be the per capita effective contact rate with pest. Hence, the rate of change in plant population is given by the following equation:

$$\frac{dJ}{dT} \;=\; R_J J(1 - \frac{J}{K_J}) - \lambda SV.$$

(2)

(A2) The infected individuals do not reproduce due to resource limitations. However, it contributes with S to population growth with carrying capacity K_S. In absence of virus (NPV), the intrinsic growth rate of the susceptible pest population cab be described in logistic fashion [17]. Let, R_J be the intrinsic birth rate and λ be the effective per capita contact rate of pest with viruses. Also pest consumes Jatropha plant, so the reproduction rate of pest is enhanced. Let, β_1 be the rate increase of reproduction of susceptible pest. Hence the differential equation of the susceptible pest is given by:

$$\frac{dS}{dT} \;=\; R_S S(1 - \frac{S+I}{K_S}) + \beta_1 JS - \lambda SV.$$

(A3) The infected class of pest is removed by lysis before having the possibility of reproducing. Let ξ_1 be the rate of mortality of infected pest and hence the growth rate of infected pest is given by the following equation:

$$\frac{dI}{dT} \;=\; \lambda SV - \xi_1 I.$$

(A4) Let κ be the rate of production of virus per pest from lysis which called virus replication parameter. Now, μ_V be the rate of mortality of virus. Also there are free virus in this environment which are reproduced constantly and we consider π_v be the constant rate of reproduction of free virus. Then the rate of change of virus is given by the following differential equation:

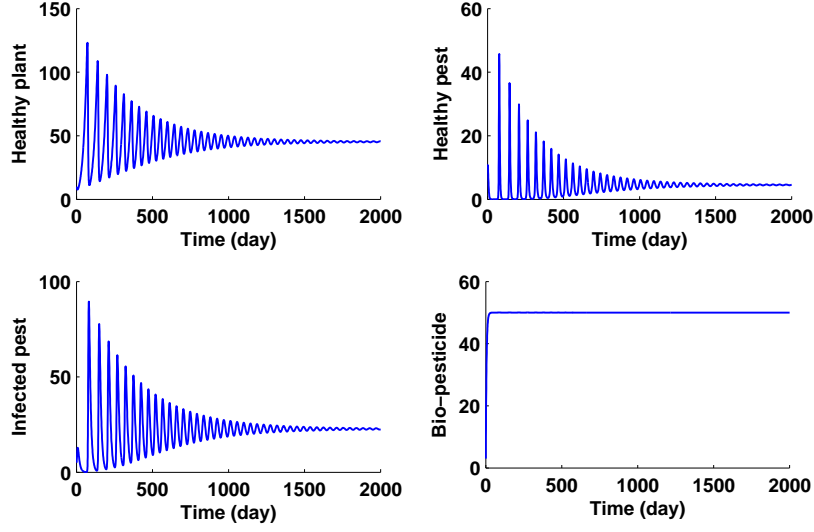$$\frac{dV}{dT} \;=\; \pi_v + \kappa \xi_1 I - \mu_V V.$$

FIGURE 1. Trajectory portrait of model system (3) with parameter value as in Table 1 and $\kappa = 0.01$.
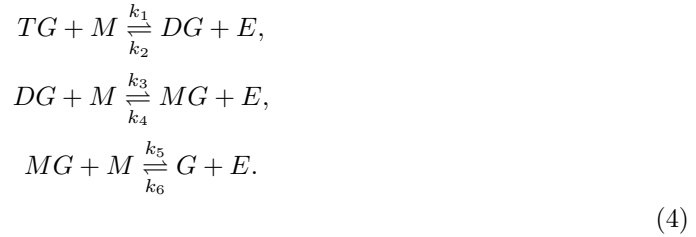
noindent Based on the above assumptions (A1) - (A4), we can further formulate the following mathematical model:

$$
\begin{aligned}
\frac{dJ}{dT} &= R_J J \left(1 - \frac{J}{K_J}\right) - \beta_1 JS, \\
\frac{dS}{dT} &= R_S S \left(1 - \frac{S+I}{K_S}\right) + \beta_1 JS - \lambda SV, \\
\frac{dI}{dT} &= \lambda SV - \xi_1 I, \\
\frac{dV}{dT} &= \pi_v + \kappa \xi_1 I - \mu_V V,
\end{aligned}
\tag{3}
$$

with initial conditions as $J(0) = J_0$, $S(0) = S_0$, $I(0) = I_0$ and $V(0) = V_0$.

3. The Mathematical Model for Biodiesel Production from Jatropha Oil

Biodiesel can be produced by the transesterification of triglycerides and methanol in the presence of an alkaline catalyst such as potassium hydroxide (KOH). The reaction consists of three steps and reversible reactions, where triglycerides (TG) is converted to diglycerides (DG), diglycerides (DG) to monoglycerides (MG) and finally monoglycerides (MG) to glycerol. The reaction steps are given below by schematic diagram:

$$
\begin{aligned}
TG + M &\underset{k_2}{\overset{k_1}{\rightleftharpoons}} DG + E, \\
DG + M &\underset{k_4}{\overset{k_3}{\rightleftharpoons}} MG + E, \\
MG + M &\underset{k_6}{\overset{k_5}{\rightleftharpoons}} G + E.
\end{aligned}
\tag{4}
$$

At each reaction step, one molecule of methyl ester is produced for each molecule of methanol consumed [18], [19].

It is shown above that the alcoholysis kinetic reaction scheme consists of three reversible reactions. Beside, there are also FFA and BD saponification reactions. It is shown below that the saponification kinetic reaction scheme consists of mainly three irreversible reactions [20] as:

$$FFA + OH \overset{k_7}{\rightarrow} A + W,$$
$$E + OH \overset{k_8}{\rightarrow} A + M. \tag{5}$$

We denote the concentrations of triglycerides, diglycerides, monoglycerides, biodiesel (methyl ester), methanol(alcohol) and glycerol by x_T, x_D, x_M, x_E, x_A and x_G respectively. Also, concentration of catalyst, fatty acid, soap and water are denoted by x_H, x_F, x_P and x_W respectively. Using the above assumption and considering the law of mass action, we get the following differential equations to characterize the stepwise reactions,

$$\begin{aligned}
\frac{dx_T}{dt} &= -k_1 x_T x_A + k_2 x_D x_E, \\
\frac{dx_D}{dt} &= k_1 x_T x_A - k_2 x_D x_E - k_3 x_D x_A + k_4 x_M x_E, \\
\frac{dx_M}{dt} &= -k_5 x_M x_A + k_6 x_G x_E + k_3 x_D x_A - k_4 x_M x_E, \\
\frac{dx_A}{dt} &= -k_1 x_T x_A + k_2 x_D x_E - k_3 x_D x_A + k_4 x_M x_E - k_5 x_M x_A + k_6 x_G x_E + k_8 x_E x_H, \\
\frac{dx_E}{dt} &= k_1 x_T x_A - k_2 x_D x_E + k_3 x_D x_A - k_4 x_M x_E + k_5 x_M x_A - k_6 x_G x_E - k_8 x_E x_H, \\
\frac{dx_G}{dt} &= k_5 x_M x_A - k_6 x_G x_E, \\
\frac{dx_F}{dt} &= -k_7 x_F x_H, \\
\frac{dx_H}{dt} &= -k_7 x_F x_H - k_8 x_E x_H, \\
\frac{dx_P}{dt} &= k_7 x_F x_H + k_8 x_E x_H, \\
\frac{dx_W}{dt} &= k_7 x_F x_H, \tag{6}
\end{aligned}$$

with the initial conditions:

$$\begin{aligned}
x_T(0) &= x_{T_0}, \ x_D(0) = 0, \ x_M(0) = 0, x_A(0) = x_{A_0}, \ x_E(0) = 0, \\
x_G(0) &= 0, x_F(0) = 0, \ x_H(0) = x_{H_0}, x_P(0) = 0 \text{ and } x_W(0) = 0. \tag{7}
\end{aligned}$$

Here k_i, (i=1, 2,...,8) are reaction rate constants. Again, the reaction constant, k_i, is expressed by the following equation:

$$k_i = a_i e^{\frac{-b_i}{T}}.$$

T is the reaction temperature in Kelvin scale (K), a_i is the frequency factor, and

$$b_i = \frac{Ea_i}{R}.$$

in which Ea_i is the activation energy for each component and R is the universal gas constant. The values of a_i and b_i are given in Table 2.

4. Mathematical Analysis of Model System (3)

4.1. Dimensionless form of the model (3)

To reduce the number of parameters and for the simplicity of analytical calculations we take the following dimensionless form of the system (3). Taking dimensionless time $t = \frac{T}{\lambda K_S}$ and using the transformations $j = \frac{J}{K_J}$, $s = \frac{S}{K_S}$, $i = \frac{I}{K_S}$ and $v = \frac{V}{K_S}$, we have the dimensionless form of the model given by

$$
\begin{aligned}
\frac{dj}{dt} &= aj(1-j) - \beta js, \\
\frac{ds}{dt} &= bs(1 - (s+i)) + \alpha js - sv, \\
\frac{di}{dt} &= sv - \xi i, \\
\frac{dv}{dt} &= v_0 + \kappa \xi i - \mu v,
\end{aligned}
\tag{8}
$$

Where $a = \frac{R_J}{\lambda K_S}$, $\beta = \frac{\beta_1}{\lambda}$, $b = \frac{R_S}{\lambda K_S}$, $\alpha = \frac{\beta_1 K_J}{\lambda K_S}$, $\xi = \frac{\xi_1}{\lambda K_S}$, $v_0 = \frac{\pi_v}{\lambda K_S^2}$, $\mu = \frac{\mu_V}{\lambda K_S}$.

4.2. Positivity and Boundedness of the system

We make an obvious assumption that all parameter used in the model are positive. The initial condition are given by,

$$
\begin{aligned}
j(0) &= j_0, \ s(0) = s_0, \ i(0) = i_0, \ v(0) = v_0, \\
&\text{with } j_0 > 0, \ s_0 > 0, \ i_0 > 0 \text{ and } v_0 > 0.
\end{aligned}
\tag{9}
$$

Now we prove the positivity and boundedness of the system by following theorem and lemma.

Theorem 4.1. *Each component of the system (8) with initial conditions (9) remains positive for all $t > 0$.*

     Proof. Let $(j(t), s(t), i(t), v(t))$ be any solution of the system (8). Now, it is clear that $j(t) > 0$ for all $t > 0$.
and

$$
\frac{ds}{dt} = bs(1 - (s+i)) + \alpha js - sv \geq -sv.
$$

Hence,

$$
s \geq s_0 e^{\int_0^t (v(t))dt} > 0, \ \text{ since } \int_0^t (v(t))dt < \infty \text{ for all } t > 0.
\tag{10}
$$

Similarly, it can prove that

$$
i \geq i_0 e^{-\xi t} > 0, \text{for all } t > 0 \text{ and } v \geq v_0 e^{-\mu t} > 0 \text{ for all } t > 0.
\tag{11}
$$

                                                                     ■

Lemma 4.2. *Define the function $H_1(t) = \alpha j(t) + \beta s(t); \ t \in [0, \infty)$. Then for all $t > 0$, $H_1(t) \leq M_1$. where, $M_1 = \frac{\alpha(a+\alpha)^2}{4a} + \frac{\beta(b+\beta)^2}{4a} + H(j(0), s(0))e^{-t}$. Hence, $j(t)$ and $s(t)$ are bounded.*

     Proof. Let $(j(t), s(t), i(t), v(t))$ be any solution of the system (8). Here $H_1(t) = \alpha j(t) + \beta s(t)$.

Therefore,

$$\begin{aligned}
\frac{dH_1}{dt} &= \alpha\frac{dj}{dt} + \beta\frac{ds}{dt} \\
&= \alpha aj(1-j) - \alpha\beta js + \beta bs(1-(s+i)) + \alpha\beta js - \beta sv \\
&\leq \alpha aj(1-j) + \beta bs(1-s) \\
&\leq \alpha aj(1+\frac{\alpha}{a}-j) + \beta bs(1+\frac{\beta}{b}-s) - H_1.
\end{aligned}$$

Hence,

$$\frac{dH_1}{dt} + H_1 \leq \alpha aj(1+\frac{\alpha}{a}-j) + \beta bs(1+\frac{\beta}{b}-s),$$

which implies that

$$H_1(j(t),s(t)) \leq \frac{\alpha(a+\alpha)^2}{4a} + \frac{\beta(b+\beta)^2}{4a} + H(j(0),s(0))e^{-t} = M_1.$$

■

Lemma 4.3. *Define the function $H_2(t) = s(t) + i(t), t \in [0,\infty)$. There exist two real number M_2 and M_3 such that $s(t) \leq M_2$ and $j(t) \leq M_3$ and we have, for all $t > 0$, $H_2(t) \leq M_4$. Where $M_4 = \frac{b+\alpha M_3}{b} + H_2(s(0),i(0))e^{-bM_2 t}$. Then $i(t)$ bounded.*

Proof. Let $(j(t),s(t),i(t),v(t))$ be any solution of the system (8). Since $j(t)$ and $s(t)$ are bounded, there exist two real number M_2 and M_3 such that $s(t) \leq M_2$ and $j(t) \leq M_3$ Since, $H_2(t) = s(t) + i(t)$. We have,

$$\begin{aligned}
\frac{dH_2}{dt} &= \frac{ds}{dt} + \frac{di}{dt} \\
&= bs(1-(s+i)) + \alpha js - sv + sv - \xi i \\
&\leq bM_2(1-(s+i)) + \alpha M_2 M_3 \\
&\leq bM_2 - bM_2 H_2 + \alpha M_2 M_3,
\end{aligned}$$

which implies,

$$\frac{dH_2}{dt} + bM_2 H_2 \leq bM_2 + \alpha M_2 M_3.$$

Hence,

$$H_2(s(t),i(t)) \leq \frac{b+\alpha M_3}{b} + H_2(s(0),i(0))e^{-bM_2 t} = M_4.$$

■

Lemma 4.4. *Let $H_3(t) = v(t), t \in [0,\infty)$. there exists a real number M_5, s.t $i(t) \leq M_5$. Then for all $t > 0$, $H_3(t) \leq M_6$, where $M_6 = \frac{v_0 + \kappa\xi M_5}{\mu} + H_3(v_0)e^{-\mu t}$. Hence $v(t)$ is bounded.*

Proof. Let $(j(t),s(t),i(t),v(t))$ be any solution of the system (8). Since $i(t)$ is bounded, there exists a real number M_5, such that $i(t) \leq M_5$, since $H_3(t) = v(t)$. We have,

$$\begin{aligned}
\frac{dH_3}{dt} &= \frac{dv}{dt} \\
&= v_0 + \kappa\xi i - \mu v \leq v_0 + \kappa\xi M_5 - \mu H_3,
\end{aligned}$$

which implies $\frac{dH_3}{dt} + \mu H_3 \leq v_0 + \kappa\xi M_5$. Hence,

$$H_3(v(t)) \leq \frac{v_0 + \kappa\xi M_5}{\mu} + H_3(v_0)e^{-\mu t} = M_6.$$

■

Theorem 4.5. *: All solution of the system (8) that start in R_+^4 are uniformly bounded.*

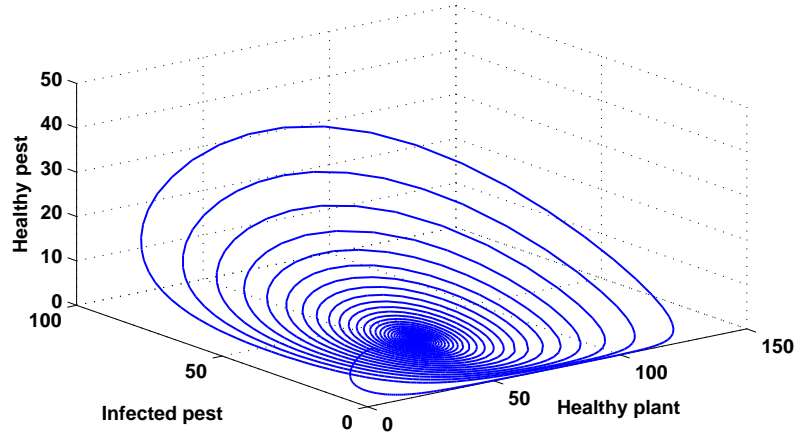The proof follows directly from Lemma 1, Lemma 2 and Lemma 3.

FIGURE 2. Phase portrait of Healthy plant $(J(t))$, Healthy pest $(S(t))$ and Virus $(V(t))$ with parameter value as in Table 1 and $\kappa = 0.01$.

4.3. Equilibria and Stability

In this sub-section we find the different equilibrium point of the dimensionless system (8) and analyze the stability of this system around these points. The above system (equation 8) has four equilibrium points, viz.

    (a) The axial equilibrium equilibrium point $E_0(0, 0, 0, \frac{v_0}{\mu})$,

    (b) Pest free equilibrium point $E_1(1, 0, 0, \frac{v_0}{\mu})$,

    (c) Virus free equilibrium point $E_2(1 - \frac{\beta}{a}, 1, 0, 0)$ and

    (d) The interior equilibrium point $E^*(j^*, s^*, i^*, v^*)$.

where,

$$j^* = 1 - \frac{\beta}{a}\frac{\mu\xi i^*}{\kappa\xi i^* + v_0},$$

$$s^* = \frac{\mu\xi i^*}{\kappa\xi i^* + v_0},$$

$v^* = \frac{\kappa\xi i^* + v_0}{\mu}$ and i^* is the positive root of the following equation:

$$
\begin{aligned}
q_1 i^{*2} + q_2 i^* + q_3 &= 0, \text{ where,} \\
q_1 &= \mu ab\kappa\xi + a\kappa^2\xi^2 \\
q_2 &= \mu^2\xi(ab + \alpha\beta) + v_0 ab\mu - \mu a\kappa\xi(b + \alpha) + a\kappa^2\xi^2 \\
q_3 &= av_0^2 - \mu\alpha av_0 - \mu abv_0.
\end{aligned}
\tag{12}
$$

For positivity of j^* implies that $i^* < \frac{av_0}{\beta\mu\xi - a\kappa\xi}$. Also for positivity of E_2 we must have $a > \beta$.

Now, the following cases may arise.

Case(I): If $\mu = \frac{v_0}{\alpha+b}$, then the coefficient q_3 vanishes and other coefficients are positive, which implies that $i^* = 0$ and consequently $s^* = 1$, $v^* = 0$ and $j^* = 1 - \frac{\beta}{a}$ i.e susceptible pest population is maximum and growth of Jatropha plant will be minimum.

Case(II): If $\mu > \frac{v_0}{\alpha+b}$, then the coefficient q_3 is negative. Also, the coefficient q_1 is always positive then for any real value of q_2, by Descartes' rule of sign, it can be said that equation (12) has exactly one positive solution i.e unique interior equilibrium E^* exists.

Case(III): Finally if, $\mu < \frac{v_0}{\alpha+b}$, then the coefficient q_3 is positive. Also, the coefficient q_1 is always positive. Now, $q_2 > 0$ if

$$(\mu - p_1)(\mu - p_2) > 0 \tag{13}$$

where,

$$
\begin{aligned}
p_1 &= \frac{a\kappa\xi(b+\alpha) - abv_0 + [(a\kappa\xi(b+\alpha) - abv_0)^2 - 8\kappa\xi^2 v_0 a(ab+\alpha\beta)]^{\frac{1}{2}}}{2\xi(ab+\alpha\beta)}, \\
p_2 &= \frac{a\kappa\xi(b+\alpha) - abv_0 - [(a\kappa\xi(b+\alpha) - abv_0)^2 - 8\kappa\xi^2 v_0 a(ab+\alpha\beta)]^{\frac{1}{2}}}{2\xi(ab+\alpha\beta)}.
\end{aligned}
\tag{14}
$$

For real and positive value of μ we must have

$$
\begin{aligned}
&\kappa\xi(b+\alpha) - abv_0 > [(a\kappa\xi(b+\alpha) - abv_0)^2 - 8\kappa\xi^2 v_0 a(ab+\alpha\beta)]^{\frac{1}{2}}, \\
\text{and} \quad &(\kappa - p_3)(\kappa - p_4) > 0.
\end{aligned}
\tag{15}
$$

Here,

$$
\begin{aligned}
p_3 &= \frac{8\xi^2 v_0 a(ab+\alpha\beta) + 2a^2 b\xi(bv_0+\alpha) + [A^2 - 4a^4 b^2 v_0^2 \xi^2(b+\alpha)^2]^{\frac{1}{2}}}{2a^2\xi^2(b+\alpha)^2}, \\
p_4 &= \frac{8\xi^2 v_0 a(ab+\alpha\beta) + 2a^2 b\xi(bv_0+\alpha) - [A^2 - 4a^4 b^2 v_0^2 \xi^2(b+\alpha)^2]^{\frac{1}{2}}}{2a^2\xi^2(b+\alpha)^2},
\end{aligned}
\tag{16}
$$

where, $A = (8\xi^2 v_0 a(ab+\alpha\beta) + 2a^2 b\xi(bv_0+\alpha))$,
For real and positive value of κ we must have the following conditions,

$$
\begin{aligned}
A^2 &> 4a^4 b^2 v_0^2 \xi^2(b+\alpha)^2 \quad \text{and} \\
A &> [A^2 - 4a^4 b^2 v_0^2 \xi^2(b+\alpha)^2]^{\frac{1}{2}}.
\end{aligned}
$$

Also, condition (13) and (15) hold if $\mu > p_1$ and $\kappa > p_3$. In this case E^* dose not exist. But when $p_1 < \frac{v_0}{\alpha+b}$ i.e when $\kappa > \frac{v_0\xi(ab+\alpha\beta) + 2abv_0(\alpha+b)}{2a\xi(\alpha+b)(2-(\alpha+b))}$, the flowing subcases may arise,

Subcase(I): When $p_1 < \mu < \frac{v_0}{\alpha+b}$, then all coefficient $q_i (i = 1, 2, 3)$ of the equation 12 are positive. This implies that E^* does not exit.

Subcase(II: When $p_2 < \mu < p_1$, then the coefficient q_2 is negative and other coefficient q_1 and q_2 are positive. So by Descartes' rule of sign the equation 12 has at least one positive root. Thus at least one positive equilibrium E^* exists.

Subcase(III): For $0 < \mu < p_2$, then all coefficient $q_i (i = 1, 2, 3)$ of the equation 12 are positive. Here E^* does not exit in that case.

Proposition 4.6. *The vanishing equilibrium point E_0 is always an unstable equilibrium point. The system is is locally asymptotically stable around the pest free equilibrium point E_1 if, $\mu < \frac{v_0}{\alpha+b}$ and the system is unstable around this point if, $\mu > \frac{v_0}{\alpha+b}$. Finally the point critically stable if, $\mu = \frac{v_0}{\alpha+b}$.*

Proof. The Jacobian matrix of the system at vanishing equilibrium point $E_0(0,0,0,\frac{v_0}{\mu})$ is given by:

$$J(0,0,0,\frac{v_0}{\mu}) = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b - \frac{v_0}{\mu} & 0 & 0 \\ 0 & \frac{v_0}{\mu} & -\xi & 0 \\ 0 & 0 & \kappa\xi & -\mu \end{bmatrix}.$$

Note that the above Jacobian matrix has at least on positive eigenvalue. Hence vanishing equilibrium point is unstable. The Jacobian matrix of the system at pest-free equilibrium point $E_1(1,0,0,\frac{v_0}{\mu})$ is given by:

$$J(0,0,0,\frac{v_0}{\mu}) = \begin{bmatrix} -a & -\beta & 0 & 0 \\ 0 & b+\alpha - \frac{v_0}{\mu} & 0 & 0 \\ 0 & \frac{v_0}{\mu} & -\xi & 0 \\ 0 & 0 & \kappa\xi & -\mu \end{bmatrix}.$$

Which gives the following characteristic equation:

$$(\lambda + a)(\lambda + \frac{v_0}{\mu} - (b+\alpha))(\lambda + \xi)(\lambda + \mu) = 0. \tag{17}$$

Here, three eigenvalues are always real and negative and other eigenvalue are given by,

$$\lambda + \frac{v_0}{\mu} - (b+\alpha) = 0. \tag{18}$$

Three cases arise here,

Case(I): When $\mu < \frac{v_0}{\alpha+b}$ then the eigenvalue is negative thus E_1 is stable.

Case(II): When $\mu > \frac{v_0}{\alpha+b}$ then the eigenvalue is positive thus E_1 is unstable.

Case(III): When $\mu = \frac{v_0}{\alpha+b}$ then the eigenvalue is zero. In this case the system is critically stable at E_1.

■

Proposition 4.7. *The virus free equilibrium point E_2 is locally asymptotically stable under $\kappa < \mu$. Whenever $\kappa = \mu$, the system enter into saddle-node bifurcation around this equilibrium point. Lastly, for $\kappa > \mu$, virus free equilibrium point E_2 is unstable.*

Proof. The Jacobian matrix of the system at virus free equilibrium point $E_2(1-\frac{\beta}{a},1,0,0)$ is given by:

$$J(1-\frac{\beta}{a},1,0,0) = \begin{bmatrix} \beta - a & \frac{beta^2}{a} - \beta & 0 & 0 \\ \alpha & \alpha(1-\frac{\beta}{a}) - b & -b & -1 \\ 0 & 0 & -\xi & 1 \\ 0 & 0 & \kappa\xi & -\mu \end{bmatrix}.$$

The characteristic equation is given by:

$$(\lambda^2 + d_1\lambda + d_2)(\lambda^2 + d_3\lambda + d_4) = 0, \tag{19}$$

where,

$$d_1 = \mu + \xi,\ d_2 = \xi(\mu - \kappa)$$
$$d_3 = a + b + \frac{\alpha\beta}{a} - (\alpha + \beta)$$
$$d_4 = (a-\beta)(b + \frac{2\alpha\beta}{a} - \alpha).$$

Now, under the conditions: $a + b + \frac{\alpha\beta}{a} > \alpha + \beta$, $a > \beta$ and $b + 2\frac{\alpha\beta}{a} > \alpha$ the following three cases are arise:

Case(I): For $\kappa < \mu$ all eigenvalue are negative, hence the virus free equilibrium point is stable.
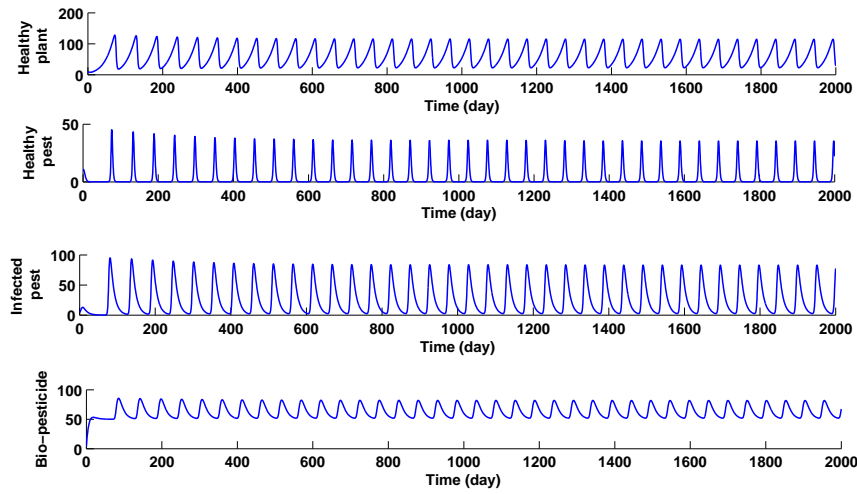
FIGURE 3. Trajectory portrait of model system (3) with $\kappa = 1$ and other parameters value as given in Table 1.
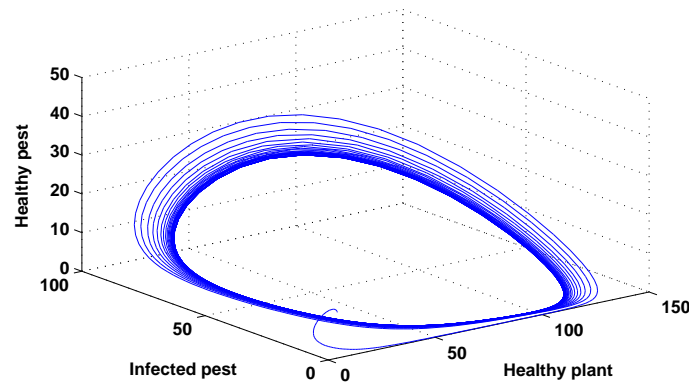

FIGURE 4. Phase portrait of Healthy plant (J(t)), Healthy pest (S(t)) and Virus (V(t)) with parameter value as in Table 1 and $\kappa = 1$.

Case(II): For $\kappa = \mu$ then one eigenvalue is zero and other three eigenvalue are negative hence the system enter into saddle-node bifurcation at E_2.

Case(III): For $\kappa > \mu$ then at least one eigenvalue is positive. Hence, the virus free equilibrium point E_2 is unstable in this case.

∎

4.4. Stability of Interior Equilibrium

The Jacobian matrix of the system at interior equilibrium point $E^*(j^*, s^*, i^*, v^*)$ is given by:

$$J(j^*, s^*, i^*, v^*) = \begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{14} \\ s_{21} & s_{22} & s_{23} & s_{24} \\ s_{31} & s_{32} & s_{33} & s_{34} \\ s_{41} & s_{42} & s_{43} & s_{44} \end{bmatrix},$$

where,

$$
\begin{aligned}
s_{11} &= a - 2aj^* - \beta s^*, s_{12} = -\beta j^*, s_{21} = \alpha s^*, \\
s_{22} &= b - 2bs^* + \alpha j^* - v^*, s_{23} = -bv^*, \\
s_{24} &= -s^*, s_{32} = v^*, s_{33} = -\xi, s_{34} = s^*, \\
s_{43} &= \kappa\xi, s_{44} = -\mu.
\end{aligned}
$$ (20)

The characteristic equation of J_{E^*} is

$$\rho^4 + \sigma_1 \rho^3 + \sigma_2 \rho^2 + \sigma_3 \rho + \sigma_4 = 0,$$ (21)

where,

$$
\begin{aligned}
\sigma_1 &= -(s_{11} + s_{22} + s_{33} + s_{44}), \\
\sigma_2 &= s_{11}s_{22} + s_{11}s_{33} + s_{11}s_{44} + s_{22}s_{33} + s_{22}s_{44} + \\
&\quad s_{33}s_{44} - s_{34}s_{43} - s_{23}s_{32} - s_{12}s_{21}, \\
\sigma_3 &= -(s_{11}s_{22}s_{33} + s_{11}s_{22}s_{44} + s_{11}s_{33}s_{44} \\
&\quad + s_{22}s_{33}s_{44}) + s_{34}s_{43}(s_{11} + s_{22}) \\
&\quad + s_{23}s_{32}(s_{11} + s_{44}) + s_{12}s_{21}(s_{33} + s_{44}) - s_{24}s_{32}s_{43}, \\
\sigma_4 &= s_{12}s_{21}s_{34}s_{43}.
\end{aligned}
$$ (22)

Then by Routh-Hurwitz criterion it follows that the interior equilibrium point locally asymptotically stable if

    (i) $\sigma_1 > 0, \sigma_4 > 0$,

    (ii) $\sigma_1 \sigma_2 - \sigma_3 > 0$ and

    (iii) $\sigma_3(\sigma_1 \sigma_2 - \sigma_3) - \sigma_1^2 \sigma_4 > 0$.

5. Mathematical Study of the System (6)

5.1. Boundedness of the System

In this section we show that the solution of the system is bounded using the following theorem.

Theorem 5.1. *All solution of the system that start in R_+^{10} are uniformly bounded.*

    Proof. We define the function $W(t)$ as follows:

$$W(t) = x_T(t) + x_D(t) + x_M(t) + x_A(t) + x_E(t) + x_G(t) + x_F(t) + x_H(t) + x_P(t) + x_W(t)$$

Therefore,

$$\frac{dW}{dt} = \frac{dx_T}{dt} + \frac{dx_D}{dt} + \frac{dx_M}{dt} + \frac{dx_A}{dt} + \frac{dx_E}{dt} + \frac{dx_G}{dt} + \frac{dx_F}{dt} + \frac{dx_H}{dt} + \frac{dx_P}{dt} + \frac{dx_W}{dt}.$$

From system (2) we have, $\frac{dW}{dt} = 0$.

Hence, $W(t) = k$, where k is a positive constant (since $x_T(0), x_A(0), x_H(0)$ are positive and others components are zero.)
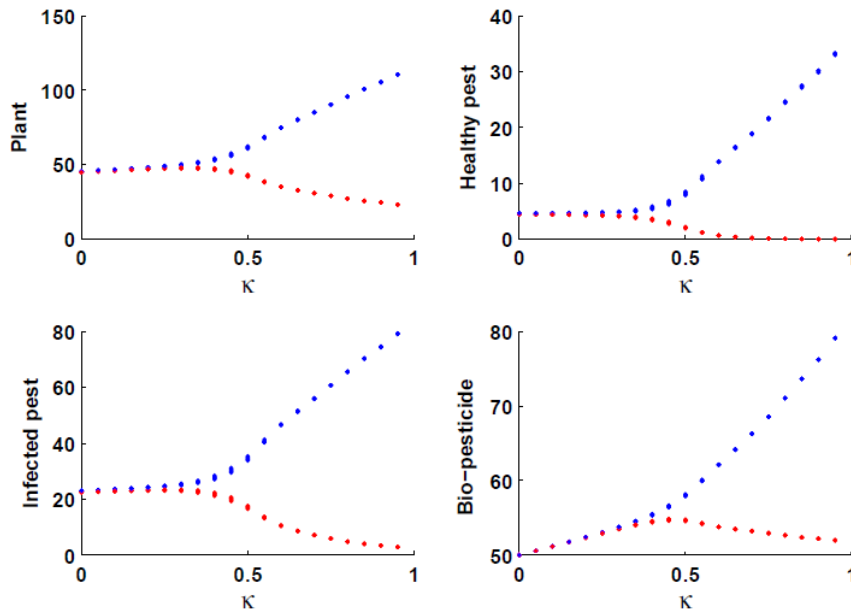
FIGURE 5. System dynamics for varying κ and parameter are as given in Table 1.

From the above analysis we have $W(t) < c$ for some $c > k$. Thus solution of the system (6) is bounded.

∎

5.2. The Optimal Control Problem for Biodiesel Production

Here we have introduced three control parameters $u_1(t)$, $u_2(t)$ to reduce soap formation and catalyst loss. They are introduced in the first stage, second stage of saponification reaction respectively. The corresponding reaction mechanism is given by the following schematic diagram

$$FFA + OH \stackrel{k_7, u_1}{\rightarrow} S + W,$$
$$BD + OH \stackrel{k_8, u_2}{\rightarrow} S + AL.$$
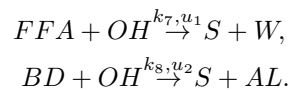
We apply the control inputs $u_1(t)$ and $u_2(t)$ to reduce the side reaction of biodiesel production to get more biodiesel in each step of transesterification reaction as quick as possible. Here, u_1, u_2 satisfy $0 \leq u_i(t) \leq 1$ [21]. Also $u_i(t) = 1$ represents the maximal use of control and $u_i(t) = 0$,

which signifies no control. Thus, introducing control input parameters, the system (6) becomes,

$$\frac{dx_T}{dt} = -k_1 x_T x_A + k_2 x_D x_E,$$

$$\frac{dx_D}{dt} = k_1 x_T x_A - k_2 x_D x_E - k_3 x_D x_A + k_4 x_M x_E,$$

$$\frac{dx_M}{dt} = -k_5 x_M x_A + k_6 x_G x_E + k_3 x_D x_A - k_4 x_M x_E,$$

$$\frac{dx_A}{dt} = -k_1 x_T x_A + k_2 x_D x_E - k_3 x_D x_A + k_4 x_M x_E - k_5 x_M x_A + k_6 x_G x_E + u_2 k_8 x_E x_H,$$

$$\frac{dx_E}{dt} = k_1 x_T x_A - k_2 x_D x_E + k_3 x_D x_A - k_4 x_M x_E + k_5 x_M x_A - k_6 x_G x_E - u_2 k_8 x_E x_H,$$

$$\frac{dx_G}{dt} = k_5 x_M x_A - k_6 x_G x_E,$$

$$\frac{dx_F}{dt} = -u_1 k_7 x_F x_H,$$

$$\frac{dx_H}{dt} = -u_1 k_7 x_F x_H - u_2 k_8 x_E x_H,$$

$$\frac{dx_P}{dt} = u_1 k_7 x_F x_H + u_2 k_8 x_E x_H,$$

$$\frac{dx_W}{dt} = u_1 k_7 x_F x_H, \tag{23}$$

with the initial conditions:

$$x_T(0) = x_{T_0}, \ x_D(0) = 0, \ x_M(0) = 0, x_A(0) = x_{A_0}, \ x_E(0) = 0,$$
$$x_G(0) = 0, x_F(0) = x_{F_0}, \ x_H(0) = 0, x_P(0) = 0 \text{and } x_W(0) = 0. \tag{24}$$

The above system can be written as:

$$\frac{dx_i}{dt} = f_i(x_i, k_i, u_1, u_2, t), i = 1, 2, ...10. \tag{25}$$

5.2.1. The Optimal system. Here we want to maximize bio-diesel x_E and minimize soap x_P, so that we define the objective cost function for the minimization problem as,

$$J(u_1, u_2) = \int_{t_i}^{t_f} [Pu_1^2(t) + Qu_2^2(t) - Rx_E^2(t) + Tx_P^2(t)]dt, \tag{26}$$

The parameters P, Q, R and T are the positive weight constants on the benefit of the cost of production. The benefit is based on the minimization of cost together with maximization of biodiesel concentration. Our aim is to find out the optimal control pair $u^* = (u_1^*, u_2^*)$ such that

$$J(u_1^*, u_2^*) = min \ (J(u_1, u_2) : (u_1, u_2) \in U),$$

where,

$$U = U_1 \times U_2 \text{ and}$$
$$U_1 = \{u_1(t) : \ u_1 \text{ is measurable and } 0 \le u_1 \le 1, \ t \in [t_i, t_f]\}$$
$$U_2 = \{u_2(t) : \ u_2 \text{ is measurable and } 0 \le u_2 \le 1, \ t \in [t_i, t_f]\}.$$

Here we use "Pontryagin Minimum Principle" [22] to find $u^*(t)$.
We define the Hamiltonian as follows:

$$H = [Pu_1^2(t) + Qu_2^2(t) - Rx_B^2(t) + Tx_S^2] + \sum \xi_i f_i, i = 1, 2, ..., 10, \tag{27}$$

where, $\xi_1, \xi_2, ..., \xi_{10}$ are adjoint variables. According to Pontryagin, adjoint variables satisfy the following equations,

$$\frac{d\xi_i}{dt} = -\frac{\partial H}{\partial x_i}, \quad i = 1, 2, ..., 10, \tag{28}$$

where $x_1 = x_T, x_2 = x_D, \cdots, x_{10} = x_W$.
This gives,

$$
\begin{aligned}
\frac{d\xi_1}{dt} &= \xi_1 k_1 x_A - \xi_2 k_1 x_A + \xi_4 k_1 x_A - \xi_5 k_1 x_A \\
\frac{d\xi_2}{dt} &= -\xi_1 k_2 x_E + \xi_2 k_2 x_E + \xi_2 k_3 x_A - \xi_3 k_3 x_A + \xi_4 k_3 x_A + \xi_5 k_2 x_E - \xi_5 k_3 x_A \\
\frac{d\xi_3}{dt} &= -\xi_2 k_4 x_E + \xi_3 k_5 x_A + \xi_3 k_4 x_E - \xi_4 k_4 x_E + \xi_4 k_5 x_A + \xi_4 k_4 x_E - \xi_4 k_5 x_A \\
&\quad -\xi_6 k_5 x_A, \\
\frac{d\xi_4}{dt} &= \xi_1 k_1 x_T - \xi_2 k_1 x_T + \xi_3 k_5 x_M - \xi_3 k_3 x_D + \xi_4 k_1 x_T + \xi_4 k_3 x_D + \xi_4 k_5 x_M \\
&\quad -\xi_5 k_1 x_T - \xi_5 k_3 x_D - \xi_5 k_5 x_M - \xi_6 k_5 x_M, \\
\frac{d\xi_5}{dt} &= 2R x_E - \xi_1 k_2 x_D + \xi_2 k_2 x_D - \xi_2 k_4 x_M - \xi_3 k_6 x_G + \xi_3 k_4 x_M - \xi_4 k_2 x_D - \xi_4 k_4 x_M \\
&\quad -\xi_4 k_6 x_G - u_2 \xi_4 k_8 x_H + \xi_5 k_2 x_D + \xi_5 k_4 x_M + \xi_5 k_6 x_G + u_2 \xi_5 k_8 x_H + \xi_6 k_6 x_G \\
&\quad +u_2 \xi_5 k_8 x_H - u_2 \xi_9 k_8 x_H \\
\frac{d\xi_6}{dt} &= -\xi_3 k_6 x_E - \xi_4 k_6 x_E + \xi_5 k_6 x_E + \xi_6 k_6 x_E \\
\frac{d\xi_7}{dt} &= u_1 \xi_7 k_7 x_H + u_1 \xi_8 k_7 x_H - u_1 \xi_9 k_7 x_H - u_1 \xi_{10} k_7 x_H \\
\frac{d\xi_8}{dt} &= u_1 \xi_7 k_7 x_F + u_1 \xi_8 k_7 x_F - u_1 \xi_9 k_7 x_F - u_1 \xi_{10} k_7 x_F - u_2 \xi_8 k_8 x_E - u_2 \xi_9 k_8 x_E \\
\frac{d\xi_9}{dt} &= -2T x_P, \\
\frac{d\xi_{10}}{dt} &= 0,
\end{aligned}
\tag{29}
$$

along with the transversality conditions $\xi_i(t_f) = 0$ for $i = 1, 2, ..., 10$.
According to Pontryagin Minimum Principle [22], the unconstrained optimal control variables u_1^* and u_2^* satisfy,

$$
\frac{\partial H}{\partial u_i^*} = 0, i = 1, 2.
\tag{30}
$$

Thus from (27) and (30), we have

$$
u_1^*(t) = \frac{k_7 x_F x_H (\xi_7 - \xi_9 + \xi_8 - \xi_{10})}{2P},
\tag{31}
$$

$$
u_2^*(t) = \frac{k_8 x_E x_H (\xi_5 - \xi_4 + \xi_8 - \xi_9)}{2Q}.
\tag{32}
$$

Due to the boundedness of the standard control we get,

$$
u_1^*(t) = max(0, \ min(1, \ \frac{k_7 x_F x_H (\xi_7 - \xi_9 + \xi_8 - \xi_{10})}{2P})),
\tag{33}
$$

In a similar way, we have the compact form of $u_2^*(t)$ as:

$$
u_2^*(t) = max(0, \ min(1, \ \frac{k_8 x_E x_H (\xi_5 - \xi_4 + \xi_8 - \xi_9)}{2Q})).
\tag{34}
$$

Thus, we have the following theorem.

Theorem 5.2. *If the objective cost function $J(u_1^*, u_2^*)$ over U is minimum for the optimal control u^* corresponding to the interior equilibrium (x_i^*, i=1, 2, ...,10), then there exist adjoint variables $\xi_1, \xi_2, ..., \xi_{10}$ which satisfy the the system of equations (29).*

TABLE 1. Values of parameters used in numerical calculation for system (3) [23, 24].

Parameters	Definition	Values (Unit)
R_J	The maximum rate of plantation	0.05 Day^{-1}
K_J	The plant density	500 M^{-2}
λ	The infection rate	0.01 Vector^{-1} Day^{-1}
β	The rete of interaction	0.05 Day^{-1}
ξ	The rate of plant loss	0.1 Day^{-1}
R_S	The growth rate of pest	0.05 Day^{-1}
β_V	The acquisition rate	0.01 Plant^{-1} Day^{-1}
K_S	The pest carrying capacity	10000 Day^{-1}

TABLE 2. Values of parameters used in numerical calculation for model system (6) [19].

Parameters	Value	Parameters	Value
a_1	3.92e7	b_1	6614.83
a_2	5.77e5	b_2	4997.98
a_3	5.88e12	b_3	9993.96
a_4	0.098e10	b_4	7366.64
a_5	5.35e3	b_5	3231.18
a_6	2.15e4	b_6	4824.87
a_7	3.92e7	b_7	6614.83
a_8	5.77e5	b_8	4997.98

6. Numerical Simulation

6.1. Numerical Simulation for Model System (3)

The dynamics of the model system are analyzed using numerical methods in MATLAB. The numerical results of the system (3) are obtained to verify the analytical predictions obtained in the previous sections.

Figure 1 shows that the model variables J(t), S(t), I(t) and V(t) oscillate initially before the system moves toward its stable region as time increases. Length of oscillation is decreasing as the time goes on. Figure 2 is the phase portrait of healthy plant, healthy pest and infected pest which shows the asymptotic stability of the system around the interior equilibrium E^* for $\kappa = 0.01$ and other parameters value as given in Table 1.

The changes in the behavior of the dynamics of the system for different values of parameter β is seen in Figure 3. Numerically, we have shown that system (3) is stable asymptotically for $\kappa < 0.3$ (approx) and when κ cross the value (0.3), the system becomes unstable and for $\kappa =1$ gives a limit cycle in (J-S-I) plane which is indicated by Figure 4.

Figure 5 depicts the bifurcation diagram of four populations to observe a perfect behavior of the system for variation of the parameter κ (the rate of interaction between Healthy Plant and Pest). We observe that when κ passes through the value 0.35 (approx.), the interior equilibrium E^* bifurcates towards a periodic solution.
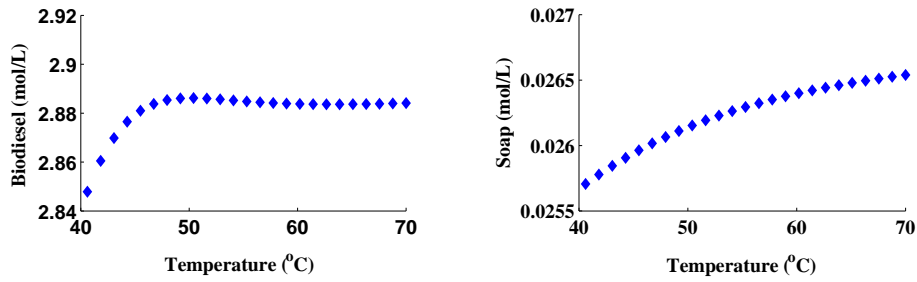
FIGURE 6. Effect of temperature on BD production and soap formation taking parameter as in Table 2 (without control).
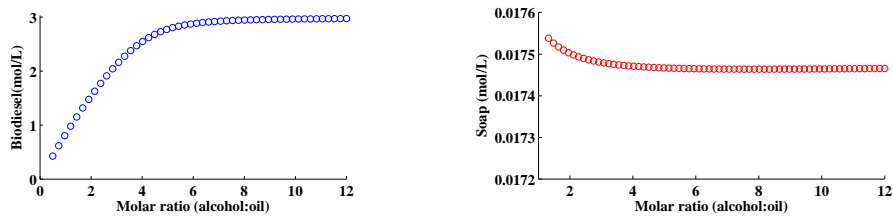

FIGURE 7. Effect of molar ratio on biodiesel production (left panel) and formation of soap (right panel) (without control).
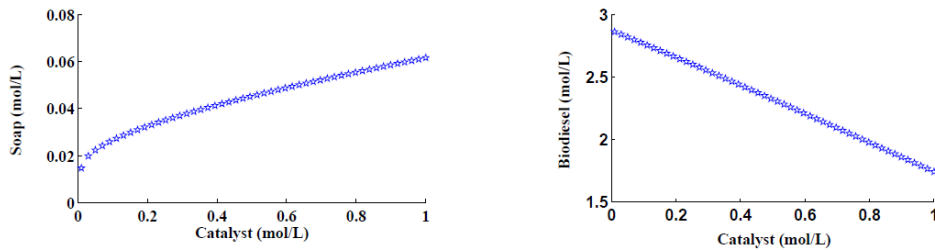

FIGURE 8. Effect of catalyst on BD production (right panel) and formation of soap (left panel) without control.

6.2. Numerical Simulation for Model System (6)

We solved the model equations (6) numerically to understand the behaviour of the transesterification reaction. The kinetics of the system has been analyzed using numerical methods in the presence and absence of the control parameters. Here, temperature, molar ratio, catalyst concentration are employed for understanding their effects in biodiesel production process.

Figure 6 shows the effect of the temperature on conversion of *Jatropha curcas* oil using 6:1 methanol to oil molar ratio with KOH concentration 1%wt of oil. It is also cleared that with
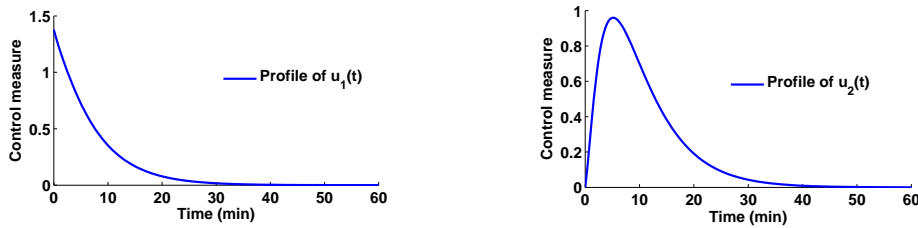
FIGURE 9. Control profiles are plotted as a function of time for biodiesel production process.

an increasing reaction temperature at or above 50^oC the initial reaction conversion from oil to biodiesel taking 1 hour reaction time, the ultimate biodiesel conversion appears to be lower compared to that of using a lower temperature. The probable reason for this is that the saponification of biodiesel and FFA by the alkali catalysts is much faster than the transesterification reaction at temperature above 50^oC. In other words, higher temperature accelerates the side saponification reaction. Thus operating the reactor under the conditions of the experiments at temperatures higher than 50^oC was not economical.

Molar ratio between alcohol and oil influences the yield of biodiesel production. Biodiesel concentration increases with the increasing ethanol/oil molar ratio. Alcohol to oil molar ratio is varied from 6:1 to 12:1 and the results are shown in figure 7. It is seen that with the increasing in the methanol to oil molar ratio, the percentage molar conversion increased rapidly with time but after 1 hour the ultimate percentage mole conversion is lower compared to that of using lower methanol to oil molar ratio.

Catalyst concentration has a significant effect on alkali-catalyzed methanolysis. We vary catalyst concentration to see the effects of catalyst concentration on the methyl ester (x_E) and soap (x_P) which is shown in Figure 8. Biodiesel yield is decreased and soap is increased as the catalyst concentrations increases. Although, the catalyst concentration of 1.5% w/w of oil provided a lower biodiesel yield than that of 1% w/w of oil for all reaction times, such a concentration should be avoided. Moreover, the methyl ester layer obtained from using this catalyst concentration has to be washed with hot distilled water several times in the water washing step. So there is a possibility of losing some biodiesel product to emulsion formation. For these reasons, 1% w/w of oil is considered to be the optimum catalyst concentration.

Figure 9 illustrates how the control policies act on the system with respect to time. From this figure we see that the control approach on each step of saponification reaction we obtain soap and if we apply control on it, then the rate of reaction monotonically decreases and soap formation reduces. We also see from this figure that the concentration of biodiesel is also enhanced. Thus, initially higher control is needed for maximum biodiesel production but after 20 minutes of the reaction, comparatively less control is needed.

In figure 10, concentration profiles of biodiesel is plotted in two different avenues which are mentioned as without control and at the optimal control level. From this figure, it is observed that how the control induced system give more production of biodiesel with respect to time. Our result shows that at 60 minute of reaction time, the concentration of biodiesel at optimal level reaches its maximum value as 2.98 mol/L, while the maximum concentration is 2.795 mol/L if
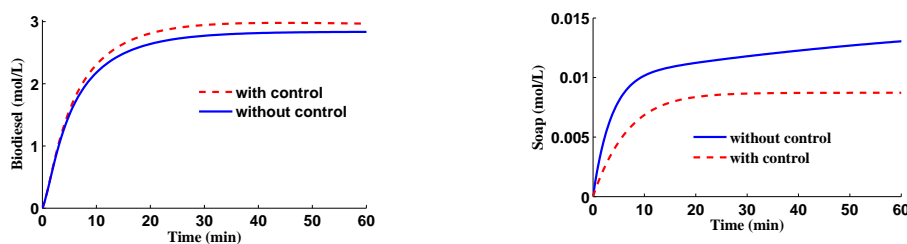
FIGURE 10. Left panel: Comparison between two cases: (1) BD concentration with optimal control, and (2) BD concentration without control; right panel:Comparison between two cases: (1) soap formation with optimal control, and (2) soap formation without control.

there is no control policy (like change of temperature, molar ratio, catalyst etc.) present on the reaction. In this way we can get 6.62% more biodiesel after 60 min of reaction time.

7. Discussion

In this article, we desire to observe the effects of biopesticide for controlling the pest of *Jatropha curcas* plant . We have explored the local stability of the positive interior equilibrium point E^* and also local Hopf-bifurcation. we have shown how the dynamics changes with the increase in the value of the parameter κ of the system. The dynamical behavior of the system is investigated from the point of view of stability and persistence. The model shows that infection can be sustained only above a threshold force of infection resulted by virus replication parameter κ. On increasing the value of κ, the endemic equilibrium bifurcates towards a periodic solution. A numerical simulation is then presented under a different choice of virus replication parameter' i.e. κ.

We have also presented another mathematical model of transesterification reaction for bio-diesel production. We have shown that the reaction depends significantly on molar ratio, catalyst concentration and temperature. Applying mathematical control approach we see that control on saponification reaction is essential in the transesterification reaction to increase bio-diesel production. By implying control approach on each step of saponification reaction, we have seen that administering the control parameters (such as temperature, catalyst concentration etc.) saponification can be reduced as well as production of bio-diesel can be increased.

8. Conclusion

The effectiveness of viral infection as a biological pest control in Jatropha plantation is discussed by mathematical modelling. It is certainly possible to eradicate pest population through release of viral pesticide. Finally, our analytical and numerical analysis indicate an important role of viral infection in pest control in *Jatropha curcas* plantation management. Thus, Application of biological pesticide for the integrated pest management policy with release of viral pesticide as biological control is the most favorable one [25]. In case of biodiesel production, fatty acid level in *Jatropha curcas* oil should be reduced as less as possible before catalytic transesterification of oil to avoid saponification problem. Also, catalyst concentration should be maintained properly. Numerical simulation offers a better understanding of optimal control for

the maximum production of biodiesel. Thus, this article provides an idea to protect Jatropha plant from different pests and gives the conditions for getting maximum biodiesel production from Jatropha oil.

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