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# ON THE REGULAR ELEMENTS OF RINGS IN WHICH THE PRODUCT OF ANY TWO ZERO DIVISORS LIES IN THE GALOIS SUBRING

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**Abstract:** Suppose $R$ is a completely primary finite ring in which the product of any two zero divisors lies in the Galois (coefficient) subring. We construct $R$ and find a generalized characterization of its regular elements.

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**Key Words:** unit groups, completely primary finite rings

## 1. Introduction

Unless otherwise stated, $J(R)$ shall denote the Jacobson radical of a completely primary finite ring $R$. We shall denote the coefficient (Galois) subring of $R$ by $R^{'}$. The set of all the regular elements in $R$ shall be denoted by $V(R)$. The rest of the notations shall be adopted from [1].

An element $x \in R$ is called regular if there exists $y \in R$ such that $x = x^2y$. The element $y$ is called a von Neumann inverse of $x$, see e.g [2]. It is well known that in any local ring, a regular element is either a unit or zero. Further details

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on the classes of completely primary finite rings considered in this work may be obtained in [3] and [4].

## 2. The Construction

Let $R^{'}$ be the Galois ring of the form $GR(p^{nr}, p^{n})$. For each $i = 1, ..., h$, let $u_i \in J(R)$, such that $U$ is an $h$- dimensional $R^{'}$-module generated by $\{u_1, ..., u_h\}$ so that $R = R^{'} \oplus U$ is an additive group. On this group, define multiplication by the following relations:

*(i)* If $n = 2$, then $u_i u_j = p\alpha_{ij}, u_i^3 = u_i^2 u_j = u_i u_j^2 = 0, u_i r^{'} = (r^{'})^{\sigma_i} u_i$

*(ii)* If $n \geq 3$, then

$$p^{n-1}u_i = 0, u_i u_j = p^2\alpha_{ij} + p^{n-1}\beta_{ij}, u_i^n = u_i^{n-1}u_j = u_i u_j^{n-1} = 0, u_i r^{'} = (r^{'})^{\sigma_i} u_i,$$

where $r^{'}, \alpha_{ij} \in R^{'}$, $\beta_{ij} \in R^{'}/pR^{'}$, $1 \leq i, j \leq h$ and $\sigma_i$ is the automorphism associated with $u_i$. Further, let $pu_i = u_i u_j = 0$, when $u_i \in U$.

From the given multiplication in $R$, we notice that $r^{'}, s^{'} \in R^{'}, \gamma_i, \lambda_i \in F_0$ are elements of $R$, then

$$(r^{'} + \sum_{i=1}^{h} \lambda_i u_i)(s^{'} + \sum_{i=1}^{h} \lambda_i u_i) = r^{'}s^{'} + p^{n-1} \sum_{i,j=1}^{h} \xi_{ij}(\lambda_i(\gamma_j)^{\sigma_i} + pR^{'})$$
$$+ \sum_{i=1}^{h} [(r^{'} + pR^{'})\gamma_i + \lambda_i(s^{'} + pR^{'})^{\sigma_i}]u_i,$$

where $r^{'}, s^{'} \in R^{'}, \lambda_i, \gamma_i \in F_0, \xi_{ij} \in R^{'}/pR^{'}$. It is easy to verify that the given multiplication turns $R$ into a ring with identity $(1, 0, ..., 0)$. We also notice that $p^{n-1} \in (J(R))^2$ when $charR = charR^{'} = p^n$, $n \geq 2$. Specifically, $p \in (J(R))^2$ when $n = 2$.

## 3. Preliminary Results

**Lemma 1.** *The ring described by the construction is commutative iff* $\sigma_i = id_{R'}$ *for each* $i = 1, ..., h$.

*Proof.* It is evident □

*Remark:* If $n = 2$, then the construction yields rings satisfying the properties

$$J(R) = pR^{'} \oplus U$$

$$(J(R))^2 = pR^{'}$$

$$(J(R))^3 = (0).$$

On the other hand, if $n \geq 3$, then $J(R) = pR^{'} \oplus U$

$$(J(R))^{n-1} = p^{n-1}R^{'}$$

$$(J(R))^n = (0).$$

Now, consider a commutative ring $R$ from the class of rings described by the construction, we notice that

$$R = R^{'} \oplus \sum_{i=1}^{h} R^{'}u_i$$

$$J(R) = pR^{'} \oplus \sum_{i=1}^{h} R^{'}u_i.$$

So

$$1 + J(R) = 1 + pR^{'} \oplus \sum_{i=1}^{h} R^{'}u_i.$$

Further, $V(R) = R^* \cup \{0\} = (R^*/1 + J(R)).(1 + J(R)) \cup \{0\} =< a > .(1 + J(R)) \cup \{0\} \cong < a > \times (1 + J(R)) \cup \{0\} \cong \mathbf{Z}_{p^r-1} \times (1 + J(R)) \cup \{0\}$. It therefore suffices to determine the structure of $1 + J(R)$.

**Proposition 1.** *For each prime integer $p$, $1+pR^{'}$ is a subgroup of $1+J(R)$.*

**Proposition 2.** *For each prime integer $p$, $1 + pR^{'} \oplus R^{'}u_1$ is a subgroup of $1 + J(R)$.*

**Proposition 3.** *For each $1 \leq j \leq h$, $1 + \sum_{j=1}^{h} \oplus R^{'}u_j$ is a subgroup of $1 + J(R)$.*

Since the two sided annihilator $ann(J(R)) = p^{n-1}R'$, we state the following result

**Proposition 4.** $1 + ann(J(R)) \leq 1 + pR' \leq 1 + J(R)$.

*Proof.* It suffices to prove that $1 + ann(J(R)) \leq 1 + pR^{'}$. Clearly $1 + ann(J(R)) = 1+p^{n-1}R^{'}, \forall n \geq 2$. Now, for $r^{'}, s^{'} \in R^{'}$, let $1+p^{n-1}r^{'}, 1+p^{n-1}s^{'} \in 1 + ann(J(R))$. Then

$$\begin{aligned}(1+p^{n-1}r^{'})(1+p^{n-1}s^{'})^{-1} &= (1+p^{n-1}r^{'})(1-p^{n-1}s^{'}) \\ &= 1+p^{n-1}(r^{'}-s^{'}) \in 1+ann(J(R)). \quad \square\end{aligned}$$

**Proposition 5.** *Let $p = 2$. Then the 2- group $1 + J(R)$ is a direct product of the subgroups $1 + pR^{'} \oplus R^{'}u_1$ by $1 + \sum_{i=1}^{h} \oplus R^{'}u_i$, with $h \geq 2$.*

**Proposition 6.** *Let $p \neq 2$. The $p-$ group $1 + J(R)$ is a direct product of the subgroups $1 + pR^{'}$ by $1 + \sum_{i=1}^{h} \oplus R^{'}u_i$.*

**Proposition 7.** *Let $U$ be a finitely generated $R^{'}-$ module. If $U$ is generated by $\{u_1, ..., u_h\}$, then $\{u_1, u_1 + u_2, ..., u_{h-1} + u_h\}$ also generates $U$.*

*Proof.* If $U$ is a finitely generated $R^{'}-$ module, then there exist $\alpha_1, ..., \alpha_h \in R^{'}$, such that every $u \in U$ can be expressed in the form $u = \sum_{i=1}^{h} \alpha_i u_i$. But $\sum_{i=1}^{h} \alpha_i u_i = (\alpha_1 - \alpha_2 + ... + (-1)^{h+1}\alpha_h)u_1 + (\alpha_2 - \alpha_3 + ... + (-1)^h\alpha_h)(u_1 + u_2) + ... + (\alpha_{h-1} - \alpha_h)(u_{h-2} + u_{h-1}) + \alpha_h(u_{h-1} + u_h)$. Since all the coefficients $\alpha_1 - \alpha_2 + ... + (-1)^{h+1}\alpha_h, \alpha_2 - \alpha_3 + ... + (-1)^h\alpha_h, ..., \alpha_{h-1} - \alpha_h$ and $\alpha_h$ belong to $R^{'}$, it follows that $\{u_1, u_1 + u_2, ..., u_{h-1} + u_h\}$ generates $U$. $\square$

**Proposition 8.** *Let $R$ be a commutative finite ring from the class of finite rings described by the construction. If $U$ is generated by $\{u_1, ..., u_h\}$, then it is also generated by $\{u_1, u_1 + u_2, ..., u_1 + u_2 + ... + u_h\}$.*

## 4. Main Results

**Proposition 9.** *Let $R$ be a commutative finite ring from the class of finite rings described by the construction. If $h \geq 1$ and $char R = p^2$, then*

$$1 + J(R) \cong \begin{cases} \boldsymbol{Z}_4^r \times (\boldsymbol{Z}_2^r)^{h-1} \text{ if } p = 2 \\ \boldsymbol{Z}_p^r \times (\boldsymbol{Z}_p^r)^h \text{ if } p \neq 2 \end{cases}$$

*Proof.* Let $\lambda_1, ..., \lambda_r \in R^{'}$ with $\lambda_1 = 1$ such that $\overline{\lambda_1}, ..., \overline{\lambda_r} \in R^{'}/pR^{'}$ form a basis for $R^{'}/pR^{'}$ regarded as a vector space over its prime subfield $GF(p)$. Since the two cases do not overlap, we treat them in turn.

*Case (i).* $p = 2$. We notice that, for every $\nu = 1, ..., r$ and $u_1 \in J(R) - (J(R))^2$,

$$(1 + \lambda_\nu u_1)^2 = 1 + 2\lambda_\nu^2 + 2\lambda_\nu u_1$$

$$= 1 + 2\lambda_\nu^2 \text{ ,since } 2 \in (J(R))^2 \text{and } 2u_1 = 0.$$

Now,

$$\begin{aligned}(1 + 2\lambda_\nu^2)(1 + \lambda_\nu u_1) &= 1 + 2\lambda_\nu^2 + (\lambda_\nu + 2\lambda_\nu^3)u_1 \\ &= 1 + 2\lambda_\nu^2 + \lambda_\nu u_1 \text{ ,since } 2 \in (J(R))^2 \text{and } 2u_1 = 0.\end{aligned}$$

But then,

$$\begin{aligned}&(1 + 2\lambda_\nu^2 + \lambda_\nu u_1)(1 + \lambda_\nu u_1) \\ &\quad = 1 + 2^2\lambda_\nu^2 + 2(\lambda_\nu + \lambda_\nu^3)u_1 \\ &\quad = 1 \text{ ,since } 2 \in (J(R))^2 \text{and } 2u_1 = 0.\end{aligned}$$

Also, for each $u_i \in J(R) - (J(R))^2, 1 \le i \le h - 1$, $(1 + \lambda_\nu u_i + \lambda_\nu u_{i+1})^2 = 1 + 2(2^2\lambda_\nu^2) + 2\lambda_\nu(u_i + u_{i+1}) = 1$, since $(J(R))^3 = (0)$ so that $2^3 = 0, 2u_i = 2u_{i+1} = 0$, as $2 \in (J(R))^2$.
So, for each $\nu = 1, ..., r$ and $1 \le i \le h - 1$, $(1 + \lambda_\nu u_1)^4 = 1$, $(1 + \sum_{i=1}^{h-1} \lambda_\nu(u_i + u_{i+1}))^2 = 1$.

For positive integers $\alpha_\nu$ , $\beta_{i\nu}$ with $\alpha_\nu \le 4$ , $\beta_{i\nu} \le 2$ $(1 \le i \le h-1, 1 \le \nu \le r)$, we notice that the equation

$$\prod_{\nu=1}^{r}\{(1 + \lambda_\nu u_1)^{\alpha_\nu}\}.\prod_{i=1}^{h-1}\prod_{\nu=1}^{r}\{(1 + \lambda_\nu(u_i + u_{i+1}))^{\beta_{i\nu}} = \{1\}$$

will imply $\alpha_\nu = 4$ and $\beta_{i\nu} = 2, 1 \le i \le h - 1$. If we set

$$T_\nu = \{(1 + \lambda_\nu u_1)^\alpha \mid \alpha = 1, ..., 4\},$$

$$S_{i\nu} = \{(1 + \lambda_\nu(u_i + u_{i+1})^{\beta_i} \mid \beta_i = 1, 2\}$$

we see that $T_\nu$, $S_{i\nu}$ are all cyclic subgroups of the group $1 + J(R)$ and they are of the orders indicated in their definition. Since

$$\prod_{\nu=1}^{r} |< 1 + \lambda_\nu u_1 >| \, . \prod_{i=1}^{h-1}\prod_{\nu=1}^{r} |< 1 + \lambda_\nu(u_i + u_{i+1}) >| = 2^{(h+1)r}$$

and the intersection of any pair of the cyclic subgroups gives the identity group, the product of the $hr$ subgroups $T_\nu$, $S_{i\nu}$, $1 \le i \le h - 1$ is direct. Therefore, their product exhausts the group $1 + J(R)$.

*Case (ii).* $p$ is odd. If $\nu = 1, ..., r$ and $u_i \in J(R) - (J(R))^2, 1 \le i \le h - 1$,

$$(1 + p\lambda_\nu)^p = 1 + p^2\lambda_\nu + \frac{p(p-1)}{2}(p\lambda_\nu)^2 + ... + (p\lambda_\nu)^p$$

$$= 1, \text{since } char R = p^2.$$

Also,

$$(1+\lambda_\nu u_1)^p = (1+\sum_{i=1}^{2}\lambda_\nu u_i)^p = ... = (1+\sum_{i=1}^{h}\lambda_\nu u_i)^p = 1.$$

For positive integers $\alpha_\nu$ , $\beta_{i\nu}$ with $\alpha_\nu \leq p$ , $\beta_{i\nu} \leq p$ $(1 \leq i \leq h, 1 \leq \nu \leq r)$, we notice that the equation

$$\prod_{\nu=1}^{r}\{(1+p\lambda_\nu)^{\alpha_\nu}\}.\prod_{i=1}^{h}\prod_{\nu=1}^{r}\{(1+\sum_{j=1}^{i}\lambda_\nu u_j)^{\beta_{i\nu}}\} = \{1\}$$

will imply $\alpha_\nu = \beta_{i\nu} = p, 1 \leq i \leq h$. If we set

$$T_\nu = \{(1+p\lambda_\nu)^\alpha \mid \alpha = 1, ..., p\},$$

$$S_{i\nu} = \{(1+\sum_{j=1}^{i}\lambda_\nu u_j)^{\beta_i} \mid \beta_i = 1, ..., p\}$$

we see that $T_\nu$, $S_{i\nu}$ are all cyclic subgroups of the group $1+J(R)$ and they are of the orders indicated in their definition. Since

$$\prod_{\nu=1}^{r} |< 1+p\lambda_\nu >| . \prod_{i=1}^{h}\prod_{\nu=1}^{r} |< 1+\sum_{j=1}^{i}\lambda_\nu u_j >| = p^{(h+1)r}$$

and the intersection of any pair of the cyclic subgroups gives the identity group, the product of the $(h+1)r$ subgroups $T_\nu$, $S_{i\nu}$, $1 \leq i \leq h$ is direct. Therefore, their product exhausts the group $1+J(R)$. $\square$

**Proposition 10.** *Let $R$ be a commutative finite ring from the class of finite rings given by the construction. If $h \geq 1, r > 1$ and $char R = p^3$, then*

$$1+J(R) \cong \begin{cases} \boldsymbol{Z}_2^r \times \boldsymbol{Z}_4^r \times (\boldsymbol{Z}_2^r)^{h-1} \text{ if } p = 2 \\ \boldsymbol{Z}_{p^2}^r \times (\boldsymbol{Z}_p^r)^h \text{ if } p \neq 2 \end{cases}$$

*Proof.* Let $\lambda_1, ..., \lambda_r \in R'$ with $\lambda_1 = 1$ such that $\overline{\lambda_1}, ..., \overline{\lambda_r} \in R'/pR'$ form a basis for $R'/pR'$ regarded as a vector space over its prime subfield $GF(p)$. We treat the two cases in turn.

*Case (i).* $p = 2$. We notice that for every $\nu = 1, ..., r$ and $u_1 \in J(R) - (J(R))^2$,

$$(-1+4\lambda_\nu)^2 = 1 - 2^3\lambda_\nu + 2^4\lambda_\nu^2$$

$$= 1, \text{ since } charR = 2^3.$$

Also

$$\begin{aligned}(1+\lambda_\nu u_1)^2 &= 1 + 2^2\lambda_\nu^2 + 2\lambda_\nu u_1 \\ &= 1 + 2^2\lambda_\nu^2 \text{ , since } 2u_1 = 0.\end{aligned}$$

But then,

$$(1 + 2^2\lambda_\nu^2)^2 = 1 + 2^3\lambda_\nu^2 + 2^4\lambda_\nu^4$$

$$= 1, \text{ since } charR = 2^3.$$

It is also easy to see that, for each $\nu = 1, ..., r$, $1 \leq i \leq h-1$, $(1+\lambda_\nu(u_i+u_{i+1}))^2 = 1$.

For positive integers $\alpha_\nu$ , $\beta_\nu$ , $\kappa_{i\nu}$ with $\alpha_\nu \leq 2$ , $\beta_\nu \leq 4$ , $\kappa_{i\nu} \leq 2, (1 \leq i \leq h-1, 1 \leq \nu \leq r)$, we notice that the equation

$$\prod_{\nu=1}^{r}\{(-1+4\lambda_\nu)^{\alpha_\nu}\}.\prod_{\nu=1}^{r}\{(1+\lambda_\nu u_1)^{\beta_\nu}\}.\prod_{i=1}^{h-1}\prod_{\nu=1}^{r}\{(1+\lambda_\nu(u_i+u_{i+1}))^{\kappa_{i\nu}} = \{1\}$$

will imply $\alpha_\nu = 2$ and $\beta_\nu = 4$, $\kappa_{i\nu} = 2, 1 \leq i \leq h-1$. If we set

$$H_\nu = \{(-1+4\lambda_\nu)^\alpha \mid \alpha = 1, 2\},$$

$$T_\nu = \{(1+\lambda_\nu u_1)^\beta \mid \beta = 1, ..., 4\},$$

$$S_{i\nu} = \{(1+\lambda_\nu(u_i+u_{i+1})^{\kappa_i} \mid \kappa_i = 1, 2\}$$

we see that $H_\nu$, $T_\nu$, $S_{i\nu}$ are all cyclic subgroups of the group $1 + J(R)$ and they are of the orders indicated in their definition. Since

$$\prod_{\nu=1}^{r} |< -1+4\lambda_\nu >| \, . \prod_{\nu=1}^{r} |< 1+\lambda_\nu u_1 >| \, . \prod_{i=1}^{h-1}\prod_{\nu=1}^{r} |< 1+\lambda_\nu(u_i+u_{i+1}) >| = 2^{(h+2)r}$$

and the intersection of any pair of the cyclic subgroups gives the identity group, the product of the $(h+1)r$ subgroups $H_\nu$, $T_\nu$, $S_{i\nu}$, $1 \leq i \leq h-1$ is direct. Therefore, their product exhausts the group $1 + J(R)$.

*Case (ii).* $p$ is odd. Here, we notice that

$$(1+p\lambda_\nu)^{p^2} = 1, (1+\lambda_\nu u_1)^p = (1+\sum_{i=1}^{2}\lambda_\nu u_i)^p = ... = (1+\sum_{i=1}^{h}\lambda_\nu u_i)^p = 1.$$

Now, for positive integers $\alpha_\nu$ , $\beta_{i\nu}$ with $\alpha_\nu \leq p^2$ , $\beta_{i\nu} \leq p$ , $(1 \leq i \leq h, 1 \leq \nu \leq r)$, we notice that the equation

$$\prod_{\nu=1}^{r}\{(1+p\lambda_\nu)^{\alpha_\nu}\}.\prod_{i=1}^{h}\prod_{\nu=1}^{r}\{(1+\sum_{j=1}^{i}\lambda_\nu u_j)^{\beta_{i\nu}}\} = \{1\}$$

will imply $\alpha_\nu = p^2$, $\beta_{i\nu} = p$ for $1 \leq \nu \leq r$ and $1 \leq i \leq h$. The rest of the proof is similar to Case (ii) in the previous proposition.

**Proposition 11.** *Let $R$ be a commutative finite ring from the class of finite rings described by the construction. If $h \geq 1$, $r = 1$ and $charR = p^n$, $n \geq 4$, then*

$$1 + J(R) \cong \begin{cases} \boldsymbol{Z}_2 \times \boldsymbol{Z}_2 \times \boldsymbol{Z}_{2^{n-2}} \times (\boldsymbol{Z}_2)^{h-1} \text{ if } p = 2 \\ \boldsymbol{Z}_{p^{n-1}} \times (\boldsymbol{Z}_p)^h \text{ if } p \neq 2 \end{cases}$$

*Proof. Case (i).* $p = 2$. Consider the element $1+2t+u_1$, where $t = n-4, n \geq 4$, then $o(1+2t+u_1) = 2^{n-2}$. The elements $-1+2^{n-1}$ and $-1+2^{n-2}+u_1$ are each of order 2. Also, the elements $1+u_1+u_2$, $1+u_2+u_3, \ldots, 1+u_{h-1}+u_h$ are each of order 2. Now, the mentioned elements generate cyclic subgroups of $1+J(R)$. Since $|< 1+2t+u_1 >| \,.\, |< -1+2^{n-1} >| \,.\, |< -1+2^{n-2}+u_1 >| \cdot \prod_{j=2}^{h} |< 1+u_{j-1}+u_j >| = 2^{n+h-1}$, and the intersection of any pair of the cyclic subgroups gives the identity group, $< 1+2t+u_1 > \times < -1+2^{n-1} > \times < -1+2^{n-2}+u_1 > \times < 1+u_1+u_2 > \times \ldots \times < 1+u_{h-1}+u_h >$ is a direct product.

*Case (ii).* $p \neq 2$ Here, the element $1+p$ is of order $p^{n-1}$ while the elements $1+u_1$, $1+\sum_{i=1}^{2} u_i, \ldots, 1+\sum_{i=1}^{h} u_i$ are each of order $p$. The given elements generate cyclic subgroups of the group $1+J(R)$. Since

$$|< 1+p >| \,.\prod_{\iota=1}^{h} |< 1+\sum_{i=1}^{\iota} u_i >| = p^{n+h-1},$$

and the intersection of any pair of the cyclic subgroups gives the identity group, $< 1+p > \times < 1+u_1 > \times < 1+\sum_{i=1}^{2} u_i > \times \ldots \times < 1+\sum_{i=1}^{h} u_i >$ is a direct product. □

**Proposition 12.** *Let $R$ be a commutative finite ring from the class of finite rings described by the construction. If $h \geq 1$, $r > 1$ and $charR = p^4$, then*

$$1 + J(R) \cong \begin{cases} \boldsymbol{Z}_2 \times \boldsymbol{Z}_2 \times \boldsymbol{Z}_4 \times \boldsymbol{Z}_2^{r-1} \times \boldsymbol{Z}_8^{r-1} \times (\boldsymbol{Z}_2^r)^{h-1} \text{ if } p = 2 \\ \boldsymbol{Z}_{p^3} \times (\boldsymbol{Z}_p^r)^h \text{ if } p \neq 2 \end{cases}$$

Let $\lambda_1, ..., \lambda_r \in R^{'}$ with $\lambda_1 = 1$ such that $\overline{\lambda_1}, ..., \overline{\lambda_r} \in R^{'}/pR^{'}$ form a basis for $R^{'}/pR^{'}$ regarded as a vector space over its prime subfield $GF(p)$. We treat the two cases in turn.

*Case (i).* $p = 2$. Clearly,

$$(-1+2^3\lambda_1)^2 = 1, (-1+2^2\lambda_1+\lambda_1 u_1)^2 = 1, (-1+2^3(\lambda_1+\lambda_2)+\lambda_2 u_1)^4 = 1,$$

$$(1+2^2(\lambda_1+\lambda_2)+\lambda_2 u_1)^2 = (1+2^2(\lambda_1+\lambda_3)+(\lambda_2+\lambda_3)u_1)^2 = ... =$$
$$(1+2^2(\lambda_1+\lambda_r)+(\lambda_2+...+\lambda_r)u_1)^2 = 1, (1+2\lambda_\nu+\lambda_\nu u_1)^8 = 1,$$
$$(1+\lambda_\nu u_{j-1}+\lambda_\nu u_j)^2 = 1, \quad 2 \leq j \leq h.$$

For positive integers $\alpha$, $\beta$, $\kappa$, $\gamma_s$, $\tau_\nu$, $\omega_{i\nu}$ with $\alpha \leq 2$, $\beta \leq 2$, $\kappa \leq 4$, $\gamma_s \leq 2$, $\tau_\nu \leq 8$, $\omega_{i\nu} \leq 2$, $2 \leq s \leq r, 1 \leq \nu \leq r, 1 \leq i \leq h-1$, we notice that the equation $\{(-1+2^3\lambda_1)^\alpha\}.\{(-1+2^2\lambda_1+\lambda_1 u_1)^\beta\}.\{(-1+2^3(\lambda_1+\lambda_2)+\lambda_2 u_1)^\kappa\}.\prod_{\nu=2}^{r}\{(1+2^2(\lambda_1+\lambda_\nu)+\sum_{\iota=2}^{\nu}\lambda_\iota u_1)^{\gamma_\nu}\}.\prod\{(1+2\lambda_\nu+\lambda_\nu u_1)^{\tau_\nu}\}.\prod_{i=1}^{h-1}\prod_{\nu=1}^{r}\{(1+\lambda_\nu(u_i+u_{i+1}))^{\omega_{i\nu}}\} = \{1\}$, will imply $\alpha = \beta = 2$, $\kappa = 4$, $\gamma_\nu = 2$, $\tau_\nu = 8$, $\omega_{i\nu} = 2$ for every $\nu = 1, ..., r$ , $\nu = 2, ..., r$ and $i = 1, ..., h-1$. If we set

$$E = \{(-1+2^3\lambda_1)^\alpha \mid \alpha = 1, 2\},$$

$$F = \{(-1+2^2\lambda_1+\lambda_1 u_1)^\beta \mid \beta = 1, 2\},$$
$$G = \{(-1+2^3(\lambda_1+\lambda_2)+\lambda_2 u_1)^\kappa \mid \kappa = 1, ..., 4\},$$
$$H_\nu = (1+2^2(\lambda_1+\lambda_\nu)+\sum_{\iota=2}^{\nu}\lambda_\iota u_1)^{\gamma_\nu} \mid \gamma_\nu = 1, 2\},$$
$$K_\nu = \{(1+2\lambda_\nu+\lambda_\nu u_1)^{\tau_\nu} \mid 1 \leq \tau_\nu \leq 8\},$$
$$L_{i\nu} = \{(1+\lambda_\nu(u_i+u_{i+1}))^{\omega_i}\}$$

we see that $E, F, G, H_2, ..., H_r, K_2, ..., K_r, L_{1\nu}, ..., L_{(h-1)\nu}$ are all cyclic subgroups of the group $1 + J(R)$ and they are of the orders indicated in their definition.

Since

$$|< -1+8\lambda_1 >| \,.\, |< -1+4\lambda_1+\lambda_1 u_1 >| \,.\, |< -1+8(\lambda_1+\lambda_2)+\lambda_2 u_1 >| \,.$$

$$\prod_{\nu=2}^{r} |< 1+4(\lambda_1+\lambda_\nu)+\sum_{\iota=2}^{\nu}\lambda_\iota u_1 >| \,.\, \prod_{\nu=2}^{r} |< 1+2\lambda_\nu+\lambda_\nu u_1 >| \,.$$

$$\prod_{i=1}^{h-1}\prod_{\nu=1}^{r} |< 1+\lambda_\nu(u_i+u_{i+1}) >| = 2^{(h+1)r},$$

and the intersection of any pair of the cyclic subgroups gives the identity group, the product of the $1+(h+1)r$ subgroups

$$E, F, G, H_2, ..., H_r, K_2, ..., K_r, L_{1\nu}, ..., L_{(h-1)\nu}$$

is direct. Therefore, their product exhausts $1+J(R)$.

*Case (ii).* $p \neq 2$. Here the proof is similar to that of Case (ii) in the previous proposition, with some slight modification. □

**Proposition 13.** *Let $R$ be a commutative finite ring from the class of finite rings described by the construction. If $h \geq 1$, $r > 1$ and $charR = p^n$, $n \geq 5$, then*

$$1+J(R) \cong \begin{cases} \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_{2^{n-2}} \times \mathbf{Z}_{2^{n-3}}^{r-1} \times \mathbf{Z}_8^{r-1} \times (\mathbf{Z}_2^r)^{h-1} \text{ if } p=2 \\ \mathbf{Z}_{p^{n-1}}^r \times (\mathbf{Z}_p^r)^h \text{ if } p \neq 2 \end{cases}$$

*Proof.* Let $\lambda_1, ..., \lambda_r \in R^{'}$ with $\lambda_1 = 1$ such that $\overline{\lambda_1}, ..., \overline{\lambda_r} \in R^{'}/pR^{'}$ form a basis for $R^{'}/pR^{'}$ regarded as a vector space over its prime subfield $GF(p)$. We treat the two cases in turn.

*Case (i).* $p=2$. Clearly,

$$(-1+2^{n-1}\lambda_1)^2 = 1, (-1+2^{n-1}\lambda_1+2^{n-1}\lambda_2)^2 = 1, (1+2\lambda_1+\lambda_1 u_1)^{2^{n-2}} = 1,$$

$(1+\sum_{\iota=2}^{\nu}\lambda_\iota u_1)^{2^{n-3}} = 1, (1+4\lambda_\nu+\lambda_\nu u_1)^{2^{n-2}} = 1, \nu = 2, ..., r, (1+\lambda_\nu(u_i+u_{i+1}))^2 = 1$, $1 \leq i \leq h-1$ For positive integers $\alpha$, $\beta$, $\kappa_\nu$, $\gamma_\nu$, $\tau_\nu$, $\omega_{i\nu}$ with $\alpha \leq 2$, $\beta \leq 2^{n-2}$, $\kappa_\nu \leq 2$, $\gamma_\nu \leq 2^{n-3}$, $\tau_\nu \leq 2^{n-2}$, $\omega_{i\nu} \leq 2$, $1 \leq \nu \leq r, 1 \leq \nu \leq r, 1 \leq i \leq h-1$ we notice that the equation $\{(-1+2^{n-1}\lambda_1)^\alpha\}.\{(1+2\lambda_1+\lambda_1 u_1)^\beta\}.\{(-1+2^{n-1}\sum_{\nu=1}^{2}\lambda_\nu)^\kappa\}.\prod_{\nu=2}^{r}\{(1+\sum_{\iota=2}^{\nu}\lambda_\iota u_1)^{\gamma_\nu}\}.\prod\{(1+4\lambda_\nu+\lambda_\nu u_1)^{\tau_\nu}\}.\prod_{i=1}^{h-1}\prod_{\nu=1}^{r}\{(1+\lambda_\nu(u_i+u_{i+1}))^{\omega_{i\nu}}\} = \{1\}$, will imply $\alpha = 2$, $\beta = 2^{n-2}$, $\kappa = 2$, $\gamma_\nu = 2^{n-3}$, $\tau_\nu = 2^{n-2}$, $\omega_{i\nu} = 2$ for every $\nu = 2, ..., r$ and $i = 1, ..., h-1$. If we set

$$E = \{(-1+2^{n-1}\lambda_1)^\alpha \mid \alpha = 1,2\}$$

$$F = \{(1+2\lambda_1+\lambda_1 u_1)^\beta \mid \beta = 1, ..., 2^{n-2}\}$$

$$G = \{(-1+2^{n-1}(\lambda_1+\lambda_2))^\kappa \mid \kappa = 1,2\}$$

$$H_\nu = (1+\sum_{\iota=2}^{\nu}\lambda_\iota u_1)^{\gamma_\nu} \mid \gamma_\nu = 1, ..., 2^{n-3}\},$$

$$K_\nu = \{(1+4\lambda_\nu+\lambda_\nu u_1)^{\tau_\nu} \mid 1, ..., 2^{n-2}\},$$

$$L_{i\nu} = \{(1+\lambda_\nu(u_i+u_{i+1}))^{\omega_i} \mid \omega_i = 1,2\}$$

we see that $E, F, G, H_2, ..., H_r, K_2, ..., K_r, L_{1\nu}, ..., L_{(h-1)\nu}$ are all cyclic subgroups of the group $1 + J(R)$ and they are of the orders indicated in their definition.

Since

$$|< -1 + 2^{n-1}\lambda_1 >| \,.\, |< 1 + 2\lambda_1 + \lambda_1 u_1 >| \,.\, \prod_{\nu=2}^{r} |< -1 + 2^{n-1}(\lambda_1 + \lambda_2) >| \,.$$

$$\prod_{\nu=2}^{r} |< 1 + \sum_{\iota=2}^{\nu} \lambda_\iota u_1 >| \,.\, \prod_{\nu=2}^{r} |< 1 + 4\lambda_\nu + \lambda_\nu u_1 >| \,.$$

$$\prod_{i=1}^{h-1} \prod_{\nu=1}^{r} |< 1 + \lambda_\nu (u_i + u_{i+1}) >| = 2^{(h+n-1)r},$$

and the intersection of any pair of the cyclic subgroups gives the identity group, the product of the $1 + (h+1)r$ subgroups $E, F, G, H_2, ..., H_r, K_2, ..., K_r, L_{1\nu}, ..., L_{(h-1)\nu}$ is direct. Therefore, their product exhausts $1 + J(R)$.

*Case (ii).* $p \neq 2$. Here the proof is similar to that of Case (ii) in the previous proposition, with some slight modification. □

We now state the main result.

**Theorem 1.** *The regular elements of the rings described by the construction is given as follows:*

*i) If* $char R = p^2$, *then*

$$V(R) \cong \begin{cases} \boldsymbol{Z}_{2^r-1} \times \boldsymbol{Z}_4^r \times (\boldsymbol{Z}_2^r)^{h-1} \cup \{0\} \text{ if } p = 2 \\ textbf{Z}_{p^r-1} \times \boldsymbol{Z}_p^r \times (\boldsymbol{Z}_p^r)^h \cup \{0\} \text{ if } p \neq 2 \end{cases}$$

*ii) If* $char R = p^3$, *then*

$$V(R) \cong \begin{cases} \boldsymbol{Z}_{2^r-1} \times \boldsymbol{Z}_2^r \times \boldsymbol{Z}_4^r \times (\boldsymbol{Z}_2^r)^{h-1} \cup \{0\} \text{ if } p = 2 \\ \boldsymbol{Z}_{p^r-1} \times \boldsymbol{Z}_{p^2}^r \times (\boldsymbol{Z}_p^r)^h \cup \{0\} \text{ if } p \neq 2 \end{cases}$$

*iii) If* $char R = p^4$, *then*

$$V(R) \cong \begin{cases} \boldsymbol{Z}_{2^r-1} \times \boldsymbol{Z}_2 \times \boldsymbol{Z}_2 \times \boldsymbol{Z}_4 \times (\boldsymbol{Z}_2)^{h-1} \cup \{0\} \text{ if } p = 2 \text{ and } r = 1 \\ \boldsymbol{Z}_{2^r-1} \times \boldsymbol{Z}_2 \times \boldsymbol{Z}_2 \times \boldsymbol{Z}_4 \times \boldsymbol{Z}_2^{r-1} \times \boldsymbol{Z}_8^{r-1} \times (\boldsymbol{Z}_2^r)^{h-1} \cup \{0\} \\ \qquad \text{if } p = 2 \text{ and } r > 1 \\ \boldsymbol{Z}_{p^r-1} \times \boldsymbol{Z}_{p^3}^r \times (\boldsymbol{Z}_p^r)^h \cup \{0\} \text{ if } p \neq 2 \end{cases}$$

*iv) If* $char R = p^n$, $n \geq 5$, *then*

$$V(R) \cong \begin{cases} \mathbf{Z}_{2^r-1} \times \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_{2^{n-2}} \times (\mathbf{Z}_2)^{h-1} \cup \{0\} & \text{if } p = 2 \text{ and } r = 1 \\ \mathbf{Z}_{2^r-1} \times \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_{2^{n-2}} \times \mathbf{Z}^{r-1}_{2^{n-3}} \times \mathbf{Z}^{r-1}_{2^{n-2}} \times (\mathbf{Z}^r_2)^{h-1} \cup \{0\} & \\ & \text{if } p = 2 \text{ and } r > 1 \\ \mathbf{Z}_{p^r-1} \times \mathbf{Z}^r_{p^{n-1}} \times (\mathbf{Z}^r_p)^h \cup \{0\} \text{ if } p \neq 2 & \end{cases}$$

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