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Automorphisms of Zero Divisor Graphs of Galois Rings

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Abstract

Let R be a commutative finite ring with unity and let $Z(R)$ be its set of zero divisors. The study of R in which the subset of zero divisors forms a unique maximal ideal has been extensively done yielding interesting and useful results. For different classes of R, the invertible element have been characterized by use of fundamental theorem of finitely generated abelian groups while $Z(R)$ has been characterized via the zero divisor graphs. Scanty in the literature are the maps that preserve the structures of R and its subsets. In this paper we discover and characterize the automorphisms of zero divisor graphs of Galois rings.

Keywords: Galois rings, zero divisors, maximal ideal

1 Introduction

The classification of the complete automorphisms of a graph is often not an easy task. Until now, the literature on the automorphisms of zero-divisor graphs is scarcely available. Much of the recent work on automorphisms of zero divisor graphs has demonstrated the fundamental importance of these graphs in the Structure theory of finite rings with identity. Most researchers have concentrated on the structure of zero divisor graphs. In this form vital parameters such as diameter, girth, binding number and connectivity of zero divisor graphs are quite conclusive. On the other hand, documented results on the automorphisms of zero divisor graphs of Galois rings are not general. For recent work on automorphisms of finite rings and unit groups of finite rings, reference can be made to [1, 2,3,4,5], Unless otherwise stated R_0 will represent Galois ring while $Z(R_0)^*$ shall denote the non-zero zero divisors, $Z(R_0)$ shall denote the Jacobson radical of R_0 and $R_0/Z(R_0)$ shall denote the Galois field of order p^r where p is prime and r be a positive integer. For any R_0 we shall denote the automorphisms of R_0 by $Aut(R_0)$ and its cardinality by $|Aut(R_0)|$. From [6] it is evident that $|R| = p^{nr}$ and $|Z(R)| = p^{(n-1)r}$ and characteristic of R is p^k. If $k = n$, R is of the form $\mathbb{Z}_{p^n}[x]/ < f >$, where f is a monic polynomial in $\mathbb{Z}_{p^n}[x]$ of degree r and irreducible modulo p . These rings are uniquely determined by the invariant p, n, r and are called Galois rings denoted by $GR(p^{nr}, p^n)$.

2 Preliminary Results

Proposition 1. *[4] Let R be a finite commutative ring. Then, there is no distinction between the left and right zero divisors and every elements is either a zero divisor or a unit.*

Proposition 2. *[7] Let R be a finite local ring. Then R is Galois if and only if $Z(R) = pR$ for some prime number p.*

Proposition 3. *[7] If R_0 is a Galois ring, then $Aut(R) \cong Aut(R/Z(R))$.*

Proposition 4. *[7] Let R be Galois ring of order p^{nr} and of characteristic p^n, having a maximal ideal $Z(R)$ such that $R/Z(R) \cong GF(p^r)$.*

Proposition 5. *[7] let R be Galois ring of the form $GR(p^{nr}, p^n)$. Then R has a unique Galois subring of the form $GR(p^{ns}, p^n)$ if and only if $s|t$.*

Next, we present results on the automorphisms of zero divisor graphs of Galois rings. Since the results are different for various characteristics of R, we present them on case to case basis.

3 Automorphisms of $\Gamma(R_0)$

Proposition 6. *Let* $Char R_0 = p$ *and* $R_0 = GR(p^r, p)$,*then* $Aut(\Gamma(R_0)) = \emptyset$

Proof. Clearly, $Z(R)^* = \emptyset$ and $V(\Gamma(R_0)) = \emptyset$. So $Aut(\Gamma(R_0))$ has no element. □

Proposition 7. *Let* $Char R_0 = p^2$ *and* $R_0 = GR(p^{2r}, p^2)$ *and* $S_0 = GR(p^2, p^2)$. *Then* $|Aut(\Gamma(R_0))| = |S_{p^r-1}| = \frac{1}{(p-2)!}|Aut(\Gamma(S_0))| \sum_{i=1}^{r} p^{r-i}(p^r - 2)!$.

Proof. The set of zero divisors, $Z(R) = pR_0$. Let pr_0 and $pr_0^{'} \in Z(R_0)$, then clearly $pr_0pr_0^{'} = p^2r_0r_0^{'} = 0$ so that $(Z(R_0))^2 = 0$ and each vertex is adjacent to the other. On the other hand $|Z(R_0)^*| = p^r - 1$ vertices, so that $\Gamma(R_0)$ induces a complete graph on $p^r - 1$ vertices. Evidently, a complete graph on $p^r - 1$ vertices has a vertex joined to each of the other $p^r - 2$ vertices by edges. Therefore an automorphism of K_{p^r-1} can map each vertex to any of the others, and in addition this does not put any limit on where any of the other $p^r - 2$ vertices are mapped, as they are all equally connected. Thus the automorphism group must be of order $(p^r - 1)(p^r - 2)(p^r - 3), \ldots (2)(1) = (p^r - 1)!$ and in particular isomorphic to S_{p^r-1}.
Now, $|Z(S_0)^*| = p - 1$. But $|Aut(\Gamma(R_0))| = (p^r - 1)! = (p - 1)(p^{r-1} + p^{r-2} + p^{r-3} + \cdots + 1) = (p - 1)\sum_{i=1}^{r} p^{r-i}(p^r - 2)!$ and $|Aut(\Gamma(S_0))| = (p - 1)!$. Dividing $|Aut(\Gamma(R_0))|$ by $|Aut(\Gamma(S_0))|$ establishes the relation $|Aut(\Gamma(R_0))| = \frac{1}{(p-2)!}|Aut(\Gamma(S_0))| \sum_{i=1}^{r} p^{r-i}(p^r - 2)!$. □

Proposition 8. *Let* R_0 *be Galois ring of order* p^{2r} *and characteristic* p^2. *Then* $|Aut(\Gamma(R_0))| = |V(\Gamma(R_0))|(p^r - 2)!$

Proof. $|Aut(\Gamma(R_0))| = (p^r-1)! = (p^r-1)(p^r-2)!$. But $|V(\Gamma(R_0)| = (p^r-1)$. Since $\frac{(p^r-1)(p^r-2)!}{(p^r-1)}$ is $(p^r - 2)!$ the results follow. □

Proposition 9. *Let* $R_0 = GR(p^{2r}, p^2)$ *and* $S_0 = GR(p^2, p^2)$. *Then* $|V(\Gamma(R_0))| = |V(\Gamma(S_0))| \sum_{i=1}^{r} p^{r-i}$.

Proof. Clearly $(Z(R_0))^2 = 0$, each zero divisor connects to the other in R_0 and S_0. But $|Z(R_0)^*| = p^r - 1$ and $|Z(S_0)^*| = p - 1$. Now $|V(\Gamma(R_0))| = (p^r - 1) = (p - 1)(p^{r-1} + p^{r-2} + p^{r-3} + \cdots + 1) = (p - 1)\sum_{i=1}^{r} p^{r-i}$ while $|V(\Gamma(S_0))| = (p - 1)$.Dividing $|V(\Gamma(R_0))|$ by $|V(\Gamma(S_0))|$ gives $|V(\Gamma(R_0))| = |V(\Gamma(S_0))| \sum_{i=1}^{r} p^{r-i}$. □

Proposition 10. *Let R_0 be a ring of order p^{2r} and characteristic p^2. Then* $|Aut(\Gamma(R_0))| = (p^r-3)!\sum deg(V(\Gamma(R_0))$.

Proof. Since $\Gamma(R_0)$ is a complete graph on p^r-1 vertices. This graph will map the p^r-1 vertices independently without any restriction giving a graph whose automorphism group is of order $(p^r-1)!$ and the sum of its degrees is $(p^r-1)(p^r-2)$. Since $\frac{(p^r-1)!}{(p^r-1)(p^r-2)}$ is $(p^r-3)!$. Evidently, the sum of the degrees divides $|Aut(\Gamma(R_0)|$ which establishes the proof. □

Proposition 11. *Let R_0 be a ring of order p^{2r} and characteristic p^2. Then* $|Aut(\Gamma(R_0))| = 2(p^r-3)!\sum E(\Gamma(R_0))$.

Proof. We note that $\Gamma(R_0)$ is a complete graph on p^r-1 vertices. So a complete graph with p^r-1 vertices has automorphism group whose order is $(p^r-1)!$ and the sum of its edges is $\frac{1}{2}(p^r-1)(p^r-2)$. Since $\frac{(p^r-1)!}{\frac{1}{2}(p^r-1)(p^r-2)}$ is $2(p^r-3)!$, the sum of the edges divides $|Aut(\Gamma(R_0)|$, which establishes the proof. □

Proposition 12. *Let $R_0 = GR(p^{2r},p^2)$ and $S_0 = GR(p^{2t},p^2)$ such that $r|t$. Then* $|Aut(\Gamma(S_0))| = \frac{|Aut(\Gamma(R_0))|(p^r-2)!\sum_{i=1}^{r}p^{r-i}}{(p^t-2)!\sum_{i=1}^{t}p^{t-i}}$

Proof. $|Aut(\Gamma(R_0))| = (p^r-1)! = (p^r-1)(p^r-2)! = (p-1)(p^{r-1}+p^{r-2}+p^{r-3}+\cdots+1) = (p-1)\sum_{i=1}^{r}p^{r-i}(p^r-2)!$. while $|Aut(\Gamma(S_0))| = (p^t-1)! = (p^t-1)(p^t-2)! = (p-1)(p^{t-1}+p^{t-2}+p^{t-3}+\cdots+1)(p^t-2)! = (p-1)\sum_{i=2}^{t}p^{t-i}(p^t-2)!$. Expressing $|Aut(\Gamma(S_0))|$ in terms of $|Aut(\Gamma(R_0))|$ gives $|Aut(\Gamma(S_0))| = \frac{|Aut(\Gamma(S_0))|(p^r-2)!\sum_{i=1}^{r}p^{r-i}}{(p^t-2)!\sum_{i=1}^{t}p^{t-i}}$ □

Proposition 13. *The automorphism group of $\Gamma(GR(p^{2r},p^2))$ with an edge removed is isomorphic to $S_2\times S_{p^r-3}$.*

Proof. Now $\Gamma(GR(p^{2r},p^2))$ is a complete graph on p^r-1 vertices. Let $\Gamma(R) = K_{p^{(r-1)}\setminus e}$, where e is an edge of K_{p^r-1}. Now $\Gamma(R)$ has a pair of vertices v and w which both have degree p^r-3 along with p^r-3 vertices all of degree p^r-2. Any automorphism permute each of these two vertices independently of the other. So the automorphism group is the direct product of two permutation groups. It is clear that the only way for the two sets of vertices is to either swap or fix the two, so this part of the direct product must be isomorphic to S_2 . Similarly, the other p^r-3 vertices all are joined to each other , so this portion of the direct product will be isomorphic to the group S_{p^r-3}. It follows therefore that the automorphism group of $\Gamma(R)$ is isomorphic to $S_2\times S_{p^r-3}$. □

Proposition 14. *Let R_0 be Galois ring of order p^n and characteristic p^n. Then $|Aut(\Gamma(R_0))| = \Pi_{l=2}^{n}(p^{n-l}(p-1))!$.*

Proof. Consider the set $X = \{p, p^2, p^3, \cdots, p^{n-1}\}$. For $p \neq 2$. Then, by [1],
$V_p = \{ps \mid (p, s) = 1\}$,
$V_{p^2} = \{p^2 s \mid (p^2, s) = 1\}$,
$V_{p^3} = \{p^3 s \mid (p^3, s) = 1\}$,
$\vdots$
$V_{p^{n-1}} = \{p^{n-1} s \mid (p^{n-1}, s) = 1\}$.
Now, we have $|V_p| = p^{n-2}(p-1)$; $|V_{p^2}| = p^{n-3}(p-1)$; $|V_{p^3}| = p^{n-4}(p-1)$ and progressing inductively, we obtain $|V_{p^{n-2}}| = p(p-1)$ and $|V_{p^{n-1}}| = (p-1)$. Then $|Aut(\Gamma(R_0))| = \Pi_{l=2}^{n}(p^{n-l}(p-1))!$. □

Proposition 15. *Let $R_0 = GR(p^{nr}, p^n)$, for $n \geq 3$ and $r \geq 1$.Then*
$|Aut(GR(p^{nr}, p^n))| = \prod_{l=2}^{n}(p^{(n-l)r}(p^r - 1))!$.

Proof. Let $\xi_1, \cdots \xi_r \in R_0$ with $\xi_1 = 1$ so that $\xi_1, \cdots \xi_r \in R_0/Z(R_0)$ form the basis for$R_0/Z(R_0)$ regarded as a vector space over its prime subfield $GF(p)$. For each prime integer p, let $X = \{p, p^2, \cdots, p^{n-1}\}$ and $V_{\Sigma a_i \xi_i}$ where $a_i \in X$, be disjoint vertices and by proposition 14, the routine enumeration yields $|Aut(GR(p^{nr}, p^n))| = \prod_{l=2}^{n}(p^{(n-l)r}(p^r - 1))!$. □

The following results is an immediate consequence of the above Proposition.

Corollary 3.1. *Let $R_0 = GR(p^{nr}, p^n)$, for $n \geq 3$ and $r \geq 1$ and $T_0 = GR(p^{2r}, p^2)$. Then $|Aut(\Gamma(R_0))| = \prod_{l=2}^{n}(p^{(n-l)r}|Aut(\Gamma(T_0))|)!$.*

Proposition 16. *Let $R_0 = GR(9, 9)$, then the Cartesian product $\Gamma(R_0) \times \Gamma(R_0)$ is a hypercube of two dimensional space. Moreover, $Aut(\Gamma(R_0) \times \Gamma(R_0)) \cong C_4$.*

Proof. Clearly, a hypercube is a Cartesian product of p edges of a complete graph on two vertices. The $GR(9, 9)$ induces a complete graph K_2 but $\Gamma(R_0) \times \Gamma(R_0) = K_2 \times K_2$ which is a hypercube of dimension two. Since $\Gamma(R_0) = K_2$ we have $\Gamma(R_0) \times \Gamma(R_0) \cong C_4$. Therefore, the automorphism group is the permutations of all the elements of C_4. □

Proposition 17. *Let $R_0 = GR(9, 9)$, then the Cartesian product $\Gamma(R_0) \times \Gamma(R_0) \times \Gamma(R_0)$ is a hypercube of three dimensional space. Furthermore, the $Aut(\Gamma(R_0)) \times \Gamma(R_0) \times \Gamma(R_0)) \cong S_4 \times C_2$.*

Proof. Clearly, a hypercube is a Cartesian product of p edges of a complete graph on two vertices. Then $GR(9, 9)$ admits a complete graph K_2 but $\Gamma(R_0) \times \Gamma(R_0) \times \Gamma(R_0) = K_2 \times K_2 \times K_2$ which is a hypercube of dimension three. The task now is to find the automorphisms of a cube. But it is well known that a cube has 24 rotational symmetries and 2 reflectional symmetries. Therefore, the automorphism group is isormorphic to $Aut(\Gamma(R_0) \times \Gamma(R_0) \times \Gamma R_0)) \cong S_4 \times C_2$. □

Proposition 18. *Let $R = GR(9, 3) \oplus \mathbb{Z}_3$, then $Aut(\Gamma(R) \times \Gamma(R))$ induces a cycle graph C_4 in which $Aut(C_4) \cong D_4$.*

Proof. We note that the most convenient representation for C_4 is precisely the same as that of D_4. Thus, any automorphism of C_4 must be the set of symmetries of D_4. Conversely, no symmetries of C_4 can be automorphism without being permutation of D_4 since graph automorphism must preserve the original structures of graph. Thus any such automorphism would be necessarily be an element of D_4. It follows that $Aut(C_4) \cong D_4$. □

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