On the Ideal Based Zero Divisor Graphs of Unital Commutative Rings and Galois Ring Module Idealizations

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

Review Article

Abstract

Let R be a commutative ring with identity 1 and I is an ideal of R. The zero divisor graph of the ring with respect to ideal has vertices defined as follows: $\{u \in I^c \mid uv \in I \text{ for some } v \in I^c\}$, where I^c is the complement of I and two distinct vertices are adjacent if and only if their product lies in the ideal. In this note, we investigate the conditions under which the zero divisor graph of the ring with respect to the ideal coincides with the zero divisor graph of the ring modulo the ideal. We also consider a case of Galois ring module idealization and investigate its ideal based zero divisor graph.

1 Introduction

Zero divisor graphs have been used in the last two decades to classify the zero divisors in rings. Interestingly, the zero divisor graphs of both commutative and non commutative rings have been studied. The concept of zero divisor graphs was discovered by Ivstan Beck in [1], where each

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element of the ring was considered to be a vertex in the graph $G(R)$. His main interest was in the colorings of the graphs. This concept was further extended by Anderson and Livingston [2] where only nonzero zero divisors were considered to be vertices of the graph $\Gamma(R)$. Later, Mulay [3] and Redmond [4] discovered compressed zero divisor graph of R and ideal based zero divisor of R respectively. Let R be a commutative ring with identity 1 and $Z(R)$ is the subset of zero divisors, so that $Z(R)^* = Z(R) - \{0\}$ is the subset of nonzero zero divisors. Conventionally, in the zero divisor graph of R, two vertices u and v are adjacent iff $uv = 0$. Let I be an ideal of R. The zero divisor graph of R with respect to I denoted by $\Gamma_I(R)$ has vertices defined as follows: $\{u \in I^c \mid uv \in I$ for some $v \in I^c\}$, where two distinct vertices u and v are adjacent iff $uv \in I$ [4]. According to Anderson and Livingston [2], $\Gamma(R)$ denotes the graph of the nonzero zero divisors in which two vertices u, v are adjacent iff $uv = 0$. Now, for the ideal I of R, R/I is a commutative quotient ring with identity $1 + I$. Two distinct vertices $u + I$ and $v + I$ are adjacent in the graph $\Gamma(R/I)$ iff $(u+I)(v+I) = I$. Some related studies are contained in [5, 6, 7, 8]. The purpose of this study is to reveal more results on the zero divisor graphs with respect to ideals and investigate the conditions under which $\Gamma_I(R)$ coincides with $\Gamma(R/I)$. Further, the ideal based zero divisor graph of a Galois-ring module idealization has been determined. Throughout the paper, (0) denotes the zero ideal, I^c denotes the complement set of I, $\sharp(V(\Gamma(R)))$ denotes the number of vertices in the zero divisor graph of R and $GR(p^{kr}, p^k)$ denotes a Galois ring of order p^{kr} and characteristic p^k where p is a prime integer, and k and r are positive integers. The rest of the notations on the theory of graphs are standard and reference may be made to [9].

2 Results

Lemma 2.1. *Let R be a commutative ring with identity 1. The adjacency $\sim$ on the graph $\Gamma(R)$ is not an equivalence relation.*

Proof. Given that R is a ring with identity, let $Z(R)$ be its subset of zero divisors so that $Z(R)^2 \neq (0)$. Clearly, the reflexive and transitive axioms fail to hold. ∎

Example:
Consider the ring $R = \mathbb{Z}_8$, of integers modulo 8. Now, $Z(R)^2 = 4\mathbb{Z}_8$. The graph $\Gamma(\mathbb{Z}_8)$ has 2, 4 and 6 as the vertices. Notice that 2 and 6 are not self adjacent. So the reflexive law fails. Further, $2 \sim 4$ and $4 \sim 6$, yet 2 and 6 are not adjacent. So the transitive law fails.

Remark: If $Z(R)^2 = (0)$, then the adjacency $\sim$ is an equivalence relation.

Now, consider the ideal $I = \{0, 4\} \subset \mathbb{Z}_8$. Then the vertex set $\Gamma_I(R) = \{2, 6\}$ and the vertices are adjacent in $\Gamma_I(R)$. But the graph of $\Gamma(R/I)$ has one vertex, viz, $2 + I$. Notice that the two graphs $\Gamma_I(R)$ and $\Gamma(R/I)$ are different in this case.

Lemma 2.2. *Let R be a commutative ring with identity 1. Suppose $I \subset R$ is an ideal, then $II^c = I^cI = I$.*

Proof. Let $x \in I$. Then $x = x.1 \in II^c$ since $1 \in I^c$. So $I \subseteq II^c$. Conversely, let $xy \in II^c$ for $x \in I$, $y \in I^c$. Then $xy \in I$, since I is an ideal. So $II^c \subseteq I$. Thus $II^c = I$. That $II^c = I^cI$ follows from commutativity of R. ∎

Remark:

$$\Gamma_{(0)}(R) = \Gamma(R) = \Gamma(R/(0))$$

Lemma 2.3. *Let R be a commutative ring with identity 1. Then $\Gamma_{(0)}(R) = \Gamma(R/(0))$*

Proof. Clearly $R/(0) = R$ and $\Gamma(R/(0)) = \Gamma(R)$ with vertices

$$\{u \in R\backslash(0) \mid uv = 0 \text{ for some } v \in R\backslash(0)\}$$

and two distinct vertices u and v are adjacent iff $uv = 0$. Obviously,

$$\{u \in R\backslash(0) \mid uv = 0 \text{ for some } v \in R\backslash(0)\} = \{u \in (0)^c \mid uv \in (0) \text{ for some } v \in (0)^c\}$$

which are the vertices of $\Gamma_{(0)}(R)$ and the adjacency of two vertices u and v is obtained when $uv \in (0)$. ∎

From Lemmata 1, 2 and 3, we obtain the following result.

Theorem 2.4. *Let R be a commutative ring with identity 1. Suppose $(0) \neq I \subset R$ is an ideal, then $\Gamma_I(R) = \Gamma(R/I)$ iff I is a maximal ideal of R.*

Proof. If I is maximal, then R/I is a field and the vertex set $V(\Gamma_I(R)) = \emptyset = V(\Gamma(R/I))$. Conversely, let $V(\Gamma_I(R)) = V(\Gamma(R/I))$, I is not maximal and $Z(R)^2 \neq (0)$. Then, there exists a proper ideal $J \subset R$ such that $I \subset J \subset R$. Now, R/I is a ring with nonzero zero divisors and $R/J \subset R/I$ so that $\sharp V(\Gamma(R/I)) > \sharp V(\Gamma(R/J))$. Next, let $u \in J\backslash I$ and $uv = vu \in I$ for some $v \in J\backslash I$, then $u, v \in \Gamma_I(R)$ and the two vertices u and v are distinct in the graph $\Gamma_I(R)$. However, $u + I = v + I$ in $\Gamma(R/I)$ so that $\sharp V(\Gamma_I(R)) > \sharp V(\Gamma(R/I))$, a contradiction. ∎

3 Ideal Based Zero Divisor Graph of a Module Idealization

Let $R_o = \text{GR}(p^{kr}, p^k)$ be a Galois Ring of order p^{kr} and characteristic p^k. Suppose U is an R_o- module generated by $u_1, u_2, ..., u_h$ so that $pu_i = 0$. Consider a $U-$ idealization $R = R_o \oplus U$ on which multiplication is defined by

$$(x_o, x_1, ..., x_h)(y_o, y_1, ..., y_h) = (x_o y_o, x_o y_1 + y_o x_1, ..., x_o y_h + y_o x_h).$$

It is easy to should that the multiplication turns R into a commutative ring with identity $(1, 0, ..., 0)$. The resultant ring is called a Galois Ring module idealization. We determine the ideal based zero divisor graph of R. It is noted that the automorphisms of R have been studied in [10], the units R^* have been studied in [11] and the graph $\Gamma(R)$ has been accounted for in [12].

The following result is Proposition 1 in [10]

Proposition 3.1. *Let $R_o = GR(p^{kr}, p^k)$ where $k \geq 3$. Then the ring R is of characteristic p^k and the subset of zero divisors $\mathcal{J}$ forms a unique maximal ideal satisfying,*
(i)$\mathcal{J} = pR_o \oplus U$
(ii) $\mathcal{J}^i = p^i R_o$
(iii)$\mathcal{J}^k = (0)$
for $i = 2, ..., k-1$.

Proof. With obvious identifications, let $R_o \subseteq R$. Since identity of R_o coincides with identity of R, it is easy to see that characteristic of R_o coincides with characteristic of R. To prove (i), it suffices to show that every element not in $\mathcal{J}$ is invertible. Let $r_o \in R_o$, and r_o is not a member of pR_o and $j \in \mathcal{J}$. Then, for a prime integer p and positive integer r,

$$(r_o + j)^{p^r} = r_o^{p^r} + j_1 \text{ (with } j_1 \in \mathcal{J})$$

$$= r_o + j_2 \text{ (with} j_2 \in J).$$

But then,

$$(r_o + j_2)^{p^r - 1} = 1 + j_3 (\text{with} j_3 \in \mathcal{J})$$

and

$$(1 + j_3)^{p^{k-1}} = 1.$$

Hence $r_o + j$ is invertible. Since

$$\mid \mathcal{J} \mid = p^{(h+k-1)r}$$

and

$$\mid (R_o/pR_o)^* + \mathcal{J} \mid = (p^r - 1)p^{(h+k-1)r},$$

it follows that $(R_o/pR_o)^* + \mathcal{J} = R - \mathcal{J}$ and hence all the elements outside $\mathcal{J}$ are invertible. Now, from the way multiplication is defined on R, it follows that

$$\mathcal{J}^i = p^i R_o$$

for $i = 2, ..., k - 1$. Specifically, when $i = k - 1$,

$$\mathcal{J}^{k-1} = p^{k-1} R_o$$

and that

$$\mathcal{J}(p^{k-1} R_o) = (p^{k-1} R_o)\mathcal{J} = (0).$$

Hence

$$\mathcal{J}^k = (0)$$

which proves (ii) and (iii) respectively. Finally, using the defined multiplication, it follows that $R\mathcal{J} = \mathcal{J}R \subseteq \mathcal{J}$, so that $\mathcal{J}$ is an ideal. Suppose there is an ideal $K \supseteq \mathcal{J}$, then K contains a unit $u \in R$ such that $uu^{-1} = u^{-1}u = 1$. Then $K = R$. Therefore, $\mathcal{J}$ is the unique maximal ideal in R since any maximal ideal distinct from $\mathcal{J}$ contains a unit. ∎

From the above result, we write the ideals

$$I_1 = pR_o \oplus U$$

$$I_2 = p^2 R_o \oplus U$$

$$\vdots$$

$$I_{k-1} = p^{k-1} R_o \oplus U$$

and

$$I_k = (0)$$

with a filtration

$$I_1 \supset I_2 \supset ... \supset I_{k-1} \supset I_k = (0)$$

The following result is a summary of the ideal based zero divisor graph of the module idealization constructed in this section.

Proposition 3.2. *Let R be the module idealization constructed in this section. Suppose $I_1, I_2, ..., I_k$ are the ideals of R defined above. Then, for $i = 1, ..., k$, $V(\Gamma_{I_i}(R)) = I_{i-1} \backslash I_i$ and two distinct vertices v and v' in $\Gamma_{I_i}(R)$ are adjacent iff $vv' \in I_i$.*

Proof. By definition, any vertex in $\Gamma_{I_i}(R)$ lies outside I_i, $1 \le i \le k$. Suppose $v \in I_j$ where $j \le i - 2$, then a typical element of I_j is of the form $p^j r_o + ru$ where $r_o, r \in R_o$ and $u \in U$. For $v' \in (I_i \cup I_{i-1})^c$, $vv' \in I_j \backslash I_i$, $1 \le i \le k$ and $1 \le j \le i - 2$. ∎

4 Conclusion

The results obtained in this paper complement those obtained by Redmond in [4]. The ideal based zero divisor graphs reduce the graph noise and may be used as building blocks for the construction of new classes of zero divisor graphs. Further research should dwell on establishing new classes of zero divisor graphs.

Competing Interests

Author has declared that no competing interests exist.

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