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On Generation of Measurable Covers for Measurable Sets Using Multiple Integral of Functions

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Abstract

In this article, we formulate an *n*-dimensional structure of measurable covers for measurable sets. Properties such as monotonocity, countable additivity and σ-finiteness of the projective tensor product of vector measure duality are largely applied.

Keywords: Measurable cover, Multiple integral, Vector measure duality.

Introduction

Over the years, vector measure integration has been used as a tool for analysis of structural properties of Banach space functions. Analysts have applied measurability concepts of the Lebesgue measure to construct measurable covers for certain subsets of the real line. In this paper, we apply multiple integral of vector valued functions with respect to projective tensor product of vector measure duality to generate measurable covers for measurable sets in the space $\Re^n$.

Basic Concepts

1. Projective Tensor Product Vector Measure Duality

Let $X_1, \ldots\ldots, X_n$ and Z be real Banach spaces with $\Phi : \Pi_{i=1}^{n} X_i \to Z$ being a continuous linear function. If $\mu_1 : R_1 \to X_1, \ldots\ldots, \mu_n : R_n \to X_n$ are countably additive vector measures, then the product

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$\Pi_{i=1}^{n}\mu_i$ is a countably additive vector measure defined on the ring $\Pi_{i=1}^{n}R_i$ generated by sets of the form $E_1 \times .... \times E_n$. Let $G(R_1),......,G(R_n)$ be a set of σ - rings generated by rings $R_1,.......,R_n$ respectively. If the extension of the vector measure $\Pi_{i=1}^{n}\mu_i : \Pi_{i=1}^{n}R_i \to \Pi_{i=1}^{n}X_i$ to a vector measure $\Pi_{i=1}^{n}\mu_i^* : \Pi_{i=1}^{n}G(R_i) \to \Pi_{i=1}^{n}X_i$ coincide with respect to a linear function $\Phi : \Pi_{i=1}^{n}X_i \to Z$, where $Z = \Phi(\mu_1^*(E_1),.........,\mu_n^*(E_n))$ for $\mu_i^*(E_i) \in X_i$, $1 \le i \le n$ and $\Pi_{i=1}^{n}X_i$ is a Banach space, then $(\Pi_{i=1}^{n})_\Phi < \mu_i^*(E_i), z' >$ is called the projective tensor product of vector measure duality between Z and itsdual space Z'.

2. Integrable Functions

An $\Pi_{i=1}^{n}G(R_i)$ - measurable function *f* is said to be integrable with respect to the projective tensor product of vector measure duality $(\Pi_{i=1}^{n})_\Phi < \mu_i^*, z' >$ if for every set $\Pi_{i=1}^{n}E_i = ((e_1,....,e_n):$ $f(e_1,....,e_n) \ne 0) \in \Pi_{i=1}^{n}G(R_i)$ there exists an element $\int_n ... \int_1 f \delta \Pi_{i=1}^{n}\mu_i^* \in \Pi_{i=1}^{n}X_i$, such that

$$< T_{(\Pi_{i=1}^{n})_\Phi \mu_i^*(E_i)}(f), Z' >= \int_n ... \int_1 f \delta (\Pi_{i=1}^{n})_\Phi < \mu_i^*(E_i), z' >$$

where Z' is the dual space of $Z = \Phi(\mu_1^*(E_1),........,\mu_n^*(E_n))$, $\mu_i^*(E_i) \in X_i$ for $1 \le i \le n$ and $z' \in Z'$, $T : L^n(\lambda) \to \Pi_{i=1}^{n}X_i$ is the operator given by $T(f) = \int_n ....\int_1 f \delta_\lambda$ for $\lambda = \mu_1^* \times ..... \times \mu_n^*$ and $f \in L^n(\lambda)$.

3. Measurable Covers

Let *f* be an integrable function with respect to the projective tensor product of vector measure duality $(\Pi_{i=1}^{n})_\Phi < \mu_i^*, z' >$. Suppose $\Pi_{i=1}^{n}E_i$ and $\Pi_{i=1}^{n}C_i$ are measurable sets with respect to the ring $\Pi_{i=1}^{n}R_i$ and σ-ring $\Pi_{i=1}^{n}G(R_i)$ respectively. We say that $\Pi_{i=1}^{n}C_i = ((c_1,.....,c_n) : f(c_1,.....,c_n) \ne 0)$ is a measurable cover of $\Pi_{i=1}^{n}E_i$ in symbols $\Pi_{i=1}^{n}E_i \Theta \Pi_{i=1}^{n}C_i$, if the following conditions are satisfied; (i) $E_i \subset C_i$ for

$i = 1,.......,n$ (ii) if $\int_n ...\int_1 f\delta(\Pi_{i=1}^n)_\Phi < \mu_i^*(A_i), z' > \leq \int_n ...\int_1 f\delta(\Pi_{i=1}^n)_\Phi < \mu_i^*(C_i - E_i), z' >$ then $A_i \downarrow \varnothing$ for each $i = 1,......n.$ Hence, $< T_{(\Pi_{i=1}^n)_\Phi \mu_i^*(A_i)}(f), Z' > = \int_n ...\int_1 f\delta(\Pi_{i=1}^n)_\Phi < \mu_i^*(A_i), z' > = 0.$ Applying the property of directed projective tensor product of vector measure duality on integrable functions, (see [1,2,12]), we write $\int_n ...\int_1 f\delta(\Pi_{i=1}^n)_\Phi < \mu_i^*(E_i), z' > = LUB_k \int_n ...\int_1 f\delta(\Pi_{i=1}^n)_\Phi < \mu_{k_i}^*(C_i), z' >$

4. Measurable Cover Estimate Technique

Let $\Pi_{i=1}^n \mu_i^* : \Pi_{i=1}^n G(R_i) \to \Pi_{i=1}^n X_i$ be a product vector measure.

Suppose $\Pi_{i=1}^n C_i = ((c_1,...,c_n) : f(c_1,...,c_n) \neq 0)$ is a set in a σ-ring $\Pi_{i=1}^n G(R_i)$ generated by a ring $\Pi_{i=1}^n R_i$. If $\Pi_{i=1}^n E_i$ is a set in $\Pi_{i=1}^n R_i$ such that $E_i \subset C_i$ for $1 \leq i \leq n$, the set $\Pi_{i=1}^n C_i$ is said to be the best approximation of a measurable cover for a measurable set $\Pi_{i=1}^n E_i$, if given a real number $\varepsilon > 0$, there exists a measurable set $\Pi_{i=1}^n A_i$, such that

$\int_n ....\int_1 f\delta\Pi_{i=1}^n < \mu_i^*(A_i), z' > \leq \int_n ....\int_1 f\delta\Pi_{i=1}^n < \mu_i^*(C_i - E_i), z' >$ and

$\int_n ....\int_1 f\delta(\Pi_{i=1}^n)_\Phi < \mu_i^*(A_i), z' > < \varepsilon$

where f is an integrable function with respect to projective tensor product of vector measure duality $(\Pi_{i=1}^n)_\Phi < \mu_i^*, z' >$ and z' is an element in the dual space Z' of Z (see [10]). In this case, the measure of $\Pi_{i=1}^n A_i$ gives the optimal error between the set $\Pi_{i=1}^n E_i$ and its cover $\Pi_{i=1}^n C_i$.

5. σ-finite Projective Tensor Product Vector Measure

A projective tensor product vector measure $\Pi_{i=1}^n \mu_i^*$ on a ring $\Pi_{i=1}^n R_i$ is said to be σ-finite if given any $\Pi_{i=1}^n E_i = N(f) \in \Pi_{i=1}^n R_i$, there exists a sequence $(\Pi_{i=1}^n E_{k_i})_{k=1}^\infty$ in $\Pi_{i=1}^n R_i$ such that $\int_n ....\int_1 f$ $\delta(\Pi_{i=1}^n)_\Phi < \mu_i^*(E_i), z' > \leq \Sigma_{k=1}^\infty \int_n ....\int_1 f \ \delta(\Pi_{i=1}^n)_\Phi < \mu_i^*(E_{k_i}), z' >$ and $\mu_i^*(E_{k_i}) < \infty$ for $1 \leq i \leq n$

Proposition 1. Let f be an integrable function with respect to $(\Pi_{i=1}^n)_\Phi < \mu_{i^*}, z' >$ such that

$$\int_n ....\int_1 f\delta(\Pi_{i=1}^n)_\Phi < \mu_{i^*}(A_i), z' > \leq \int_n ....\int_1 f\delta(\Pi_{i=1}^n)_\Phi < \mu_{i^*}(E_i), z' >$$

$$\leq \int_n ....\int_1 f\delta(\Pi_{i=1}^n)_\Phi < \mu_{i^*}(C_i), z' >$$

where E_i and C_i are sets in R_i and $G(R_i)$ respectively for $1 \leq i \leq n$. If A_i is measurably covered by C_i for $1 \leq i \leq n$, then $\Pi_{i=1}^n A_i \ \Theta \ \Pi_{i=1}^n E_i$,

Proof. Let $\Pi_{i=1}^n C_i = ((c_1, ..., c_n) : f(c_1, ..., c_n) \neq 0) \in \Pi_{i=1}^n G(R_i)$. From the hypothesis, we have

$$\int_n ....\int_1 f\delta(\Pi_{i=1}^n)_\Phi < \mu_i^*(E_i - A_i), z' > \leq \int_n ....\int_1 f\delta(\Pi_{i=1}^n)_\Phi < \mu_i^*(Ci - A_i), z' >$$

Since A_i is measurably covered by C_i for $1 \leq i \leq n$, it follows that $< T_{(\Pi_{i=1}^n)_\Phi \mu_i^*(E_i - A_i)}(f), Z' >= 0.$

By integrability of f (see [4,5]), we have $\int_n ....\int_1 f\delta(\Pi_{i=1}^n)_\Phi < \mu_i^*(E_i - A_i), z' >= 0$. Suppose $\int_n ....\int_1 f\delta(\Pi_{i=1}^n)_\Phi < \mu_i^*(A_{i_k}), z' > \leq \int_n ....\int_1 f\delta(\Pi_{i=1}^n)_\Phi < \mu_i^*(E_i - A_i), z' >$, where $A_{i_k} \neq A_i$ for $i_k \neq i < T_{(\Pi_{i=1}^n)_\Phi \mu_i^*(A_{k_i})}(f), Z' >= 0.$ As is the case with the integral operator T_f with respect to projective tensor product of vector measure duality (see[9, p. 485]). So $\int_n ....\int_1 f\delta(\Pi_{i=1}^n)_\Phi < \mu_i^*(A_{i_k}), z' >= 0$

$\Rightarrow \Pi_{i=1}^n A_i \Theta \Pi_{i=1}^n E_i$.

Proposition 2. Let $\Pi_{i=1}^n C_i = ((c_1, ..., c_n) : f(c_1, ..., c_n) \neq 0)$ If a set $(\Pi_{i=1}^n E_i)$ is measurably covered by $(\Pi_{i=1}^n C_i)$ such that $\Pi_{i=1}^n E_{i_k} \uparrow \Pi_{i=1}^n E_i$ and $\Pi_{i=1}^n C_{i_k} \uparrow \Pi_{i=1}^n C_i$, then $\cup_{k=1}^\infty \Pi_{i=1}^n E_{i_k} \Theta \cup_{k=1}^\infty \Pi_{i=1}^n C_{i_k}$

Proof. Since $\Pi_{i=1}^n E_{i_k} \uparrow \Pi_{i=1}^n E_i$ and $\Pi_{i=1}^n C_{i_k} \uparrow \Pi_{i=1}^n C_i$, it shown in [8], that $\cup_{k=1}^\infty \Pi_{i=1}^n E_{i_k} = \Pi_{i=1}^n E_i$ and $\cup_{k=1}^\infty \Pi_{i=1}^n C_{i_k} = \Pi_{i=1}^n C_i$. Let $\cup_{k=1}^\infty \Pi_i^n A_{i_k}$ be a measurable set with respect to $\Pi_{i=1}^n G(R_i)$ such that

$$\int_n ......\int_1 f\delta(\Pi_{i=1}^n)_\Phi < \mu_i^*(\cup_{k=1}^\infty A_{i_k}), z' > \leq \int_n ......\int_1 f\delta(\Pi_{i=1}^n)_\Phi < \mu_i^*(\cup_{k=1}^\infty (C_{i_k} - E_{i_k}), z' >.$$

It is shown in [12] that each integrable function *f* with respect to vector measure duality can be identified with the operator *T*. Therefore, we need to show that $\Sigma_{k=1}^{\infty} < T_{(\Pi_i^n)_\Phi \mu_i^*(A_{i_k})}(f), Z' >= 0$.

$$\Sigma_{k=1}^{\infty}\int_n ....\int_1 f\delta(\Pi_{i=1}^n)_\Phi < \mu_i^*(A_{k_i}), z' >\leq \int_n ....\int_1 f\delta(\Pi_{i=1}^n)_\Phi < \mu_i^*(C_{k_i} - \cup_{k=1}^{\infty} E_{i_k}), z' >$$

$\leq \int_n ....\int_1 f\delta(\Pi_{i=1}^n)_\Phi < \mu_i^*(\cup_{k=1}^{\infty}(C_{k_i} - E_{k_i}), z' >$. Under countable additivity, the result in [13] establishes that

$$\Sigma_{k=1}^{\infty}\int_n ....\int_1 f\delta(\Pi_{i=1}^n)_\Phi < \mu_i^*(A_{k_i}), z' >\leq \int_n ....\int_1 f\delta(\Pi_{i=1}^n)_\Phi < \mu_i^*(\cup_{k=1}^{\infty}(C_{k_i} - E_{k_i}), z' >$$

$$\leq \Sigma_{k=1}^{\infty}\int_n ....\int_1 f\delta(\Pi_{i=1}^n)_\Phi < \mu_i^*((C_{k_i} - E_{k_i}), z' >= \int_n ....\int_1 f\delta(\Pi_i^n)_\Phi < \mu_i^*(C_i - E_i), z' >$$

By hypothesis $\Pi_{i=1}^n E_i \Theta \Pi_{i=1}^n C_i$. So,

$$\Sigma_{k=1}^{\infty}\int_n ....\int_1 f\delta(\Pi_{i=1}^n)_\Phi < \mu_i^*(A_{i_k}), z' >= 0 < T_{(\Pi_i^n)_\Phi \mu_i^*(\cup_{k=1}^{\infty} A_{i_k})}(f), Z' >= 0$$

$$\Rightarrow \Sigma_{k=1}^{\infty} < T_{(\Pi_i^n)_\Phi \mu_i^*(A_{i_k})}(f), Z' >= 0$$

Proposition 3. If $\Pi_{i=1}^n A_i \Theta \Pi_{i=1}^n E_i$ and $\Pi_{i=1}^n A_i \Theta \Pi_{i=1}^n C_i = N(f)$, such that

$\int_n ....\int_1 f\delta(\Pi_{i=1}^n)_\Phi < \mu_i^*(E_i \Delta C_i), z' >= \int_n ....\int_1 f\delta(\Pi_{i=1}^n)_\Phi < \mu_i^*((E_i - C_i)\cup(C_i - E_i)), z' >$, then $< T_{(\Pi_{i=1}^n)_\Phi \mu_i^*(E_i \Delta C_i)}(f), Z' >= 0$ and therefore,

$$\int_n ....\int_1 f\delta(\Pi_{i=1}^n)_\Phi < \mu_i^*(E_i), z' >= \int_n ......\int_1 f\delta(\Pi_{i=1}^n)_\Phi < \mu_i^*(C_i), z' >$$
$$= \int_n ....\int_1 f\delta(\Pi_{i=1}^n)_\Phi < \mu_i^*(E_i \cup C_i), z' >$$
$$= \int_n ....\int_1 f\delta(\Pi_{i=1}^n)_\Phi < \mu_i^*(E_i \cap C_i), z' >.$$

Proof. Since $\Pi_{i=1}^n A_i \Theta \Pi_{i=1}^n E_i$ and $\Pi_{i=1}^n A_i \Theta \Pi_{i=1}^n C_i$, by theorem on monotone classes (see [7] p.13), it follows that

$$\int_n ....\int_1 f\delta(\Pi_{i=1}^n)_\Phi < \mu_i^*(A_i), z' >\leq \int_n ....\int_1 f\delta(\Pi_{i=1}^n)_\Phi < \mu_i^*(E_i \cap C_i), z' >$$

$$\leq \int_n ....\int_1 f\delta(\Pi_{i=1}^n)_\varphi < \mu_i^*(E_i), z' >.$$

Therefore,

$$\int_n ....\int_1 f\delta(\Pi_{i=1}^n)_\Phi < \mu_i^*(E_i - (E_i \cap C_i)), z' >\leq \int_n ....\int_1 f\delta(\Pi_{i=1}^n)_\Phi < \mu_i^*(E_i - A_i), z' >.$$

Since $\Pi_{i=1}^{n} A_i \Theta \Pi_{i=1}^{n} E_i$,then by hypothesis,

$$< T_{(\Pi_{i=1}^{n})_{\Phi}\mu_i^*(E_i)}(f), Z' > - < T_{(\Pi_{i=1}^{n})_{\Phi}\mu_i^*(E_i \cap C_i)}(f), Z' >$$

$$=< T_{(\Pi_{i=1}^{n})_{\Phi}\mu_i^*(E_i - E_i \cap F_i)}(f), Z' >= 0$$

(see [10], Th.14, p.485)

It follows that

$$\int_n \ldots\ldots \int_1 f\delta(\Pi_{i=1}^{n})_{\Phi} < \mu_i^*(E_i - E_i \cap C_i), z' >= 0$$

so that

$$\int_n \ldots\ldots \int_1 f\delta(\Pi_{i=1}^{n})_{\Phi} < \mu_i^*(E_i - C_i), z' >= 0.$$

Similarly, since

$$\int_n \ldots\ldots \int_1 f\delta(\Pi_{i=1}^{n})_{\Phi} < \mu_i^*(A_i), z' >\leq \int_n \ldots\ldots \int_1 f\delta(\Pi_{i=1}^{n})_{\Phi} < \mu_i^*(E_i \cap C_i), z' >$$

$$\leq \int_n \ldots\ldots \int_1 f\delta(\Pi_{i=1}^{n})_{\Phi} < \mu_i^* C_i, z' > .$$

So $\int_n \ldots\ldots \int_1 f\delta(\Pi_{i=1}^{n})_{\Phi} < \mu_i^*(C_i - (E_i \cap C_i), z' >\leq \int_n \ldots\ldots \int_1 f\delta(\Pi_{i=1}^{n})_{\Phi} < \mu_i^*(C_i - A_i), z' > .$

By hypothesis, $\Pi_{i=1}^{n} A_i \Theta \Pi_{i=1}^{n} C_i \Rightarrow < T_{(\Pi_{i=1}^{n})_{\Phi}\mu_i^*(C_i - E_i \cap C_i)}(f), Z' >= 0.$

As a consequence of the integral operator T acting on the integrable function f, (see [11], p. 34), we obtain

$$\int_n \ldots\ldots \int_1 f\delta(\Pi_{i=1}^{n})_{\Phi} < \mu_i^*(C_i - E_i \cap C_i), z' >= 0 \Rightarrow \int_n \ldots\ldots \int_1 f\delta(\Pi_{i=1}^{n})_{\Phi} < \mu_i^*(C_i - E_i), z' >$$

$$= 0 \int_n \ldots\ldots \int_1 f\delta(\Pi_{i=1}^{n})_{\Phi} < \mu_i^*(E_i \Delta C_i), z' >= 0.$$

Now, consider the relation

$$\int_n \ldots\ldots \int_1 f\delta(\Pi_{i=1}^{n})_{\Phi} < \mu_i^*(E_i \cup C_i), z' >= \int_n \ldots\ldots \int_1 f\delta(\Pi_{i=1}^{n})_{\Phi} < \mu_i^*((E_i \Delta C_i) \cup (E_i \cap C_i)), z' > .$$

Applying results obtained in [6, 11] on vector measure additivity and the symmetric difference of measurable sets (see [3], p.3), we obtain

$$\int_n \ldots\ldots \int_1 f\delta(\Pi_{i=1}^{n})_{\Phi} < \mu_i^*(E_i \cup C_i), z' >= \int_n \ldots\ldots \int_1 f\delta(\Pi_{i=1}^{n})_{\Phi} < \mu_i^*(E_i \Delta C_i), z' >$$

$$+ \int_n \ldots\ldots \int_1 f\delta(\Pi_{i=1}^{n})_{\Phi} < \mu_i^*(E_i \cap C_i)), z' > \int_n \ldots\ldots \int_1 f\delta(\Pi_{i=1}^{n})_{\Phi} < \mu_i^*(E_i \cup C_i), z' >$$

$$= \int_n \ldots\ldots \int_1 \delta(\Pi_{i=1}^{n})_{\Phi} < \mu_i^*(E_i \cap C_i), z' > .$$

Since $\Pi_{i=1}^{n}(E_i \cap C_i) \subset \Pi_{i=1}^{n} E_i \subset \Pi_{i=1}^{n}(E_i \cup C_i)$ and

$$\Pi_{i=1}^{n}(E_i \cap C_i) \subset \Pi_{i=1}^{n} C_i \subset \Pi_{i=1}^{n}(E_i \cup C_i) \int_n \ldots\ldots \int_1 f\delta(\Pi_{i=1}^{n})_{\Phi} < \mu_i^*(E_i \cap C_i), z' >$$
$$= \int_n \ldots\ldots \int_1 f\delta(\Pi_{i=1}^{n})_{\Phi} < \mu_i^*(E_i), z' >$$
$$= \int_n \ldots\ldots \int_1 f\delta(\Pi_{i=1}^{n})_{\Phi} < \mu_i^*(C_i), z' >$$
$$= \int_n \ldots\ldots \int_1 f\delta(\Pi_{i=1}^{n})_{\Phi} < \mu_i^*(E_i \cup C_i), z' >$$

Proposition 4. Let $\Pi_{i=1}^{n} E_i = ((e_1, \ldots.. e_n) : f((e_1, \ldots.. e_n) \neq 0)$ be a measurable set with respect to $\Pi_{i=1}^{n} R_i$. If $\int_n \ldots\ldots \int_1 f\delta(\Pi_{i=1}^{n})_{\Phi} < \mu_i^*(E_i), z' >< \infty$, there exists a set $\Pi_{i=1}^{n} C_i \in \Pi_{i=1}^{n} G(R_i)$ such that

$$\int_n \ldots\ldots\ldots \int_1 f\delta(\Pi_{i=1}^{n})_{\Phi} < \mu_i^*(E_i), z' >= LUB_k \int_n \ldots\ldots\ldots. \int_1 f\delta(\Pi_{i=1}^{n})_{\Phi} < \mu_{k_i}^*(C_i), z' >$$

Proof.. Choose a set $(A_i, i = 1, 2\ldots. n)$ in $\Pi_{i=1}^{n} G(R_i)$ such that

$$\int_n \ldots\ldots \int_1 f\delta(\Pi_{i=1}^{n})_{\Phi} < \mu_i^*(E_i), z' >\leq \int_n \ldots\ldots \int_1 f\delta(\Pi_{i=1}^{n})_{\Phi} < \mu_i^*(A_i), z' > N(f) = \Pi_{i=1}^{n} E_i.$$

by hypothesis, $\chi_{\Pi_{i=1}^{n} E_i} f \leq f$ for $f > 0 \Rightarrow \chi_{\Pi_{i=1}^{n} E_i} f \leq \chi_{\Pi_{i=1}^{n} A_i} f$. On applying the directed projective tensor product of vector measure duality, we obtain

$LUB_k \int_n \ldots. \int_1 f\delta(\Pi_{i=1}^{n})_{\Phi} < \mu_{k_i}^*(A_i), z' >\downarrow \int_n \ldots. \int_1 f\delta(\Pi_{i=1}^{n})_{\Phi} < \mu_i^*(E_i), z' >$ for each *k* (see [7], p.20), and $\int_n \ldots\ldots \int_1 f\delta(\Pi_{i=1}^{n})_{\Phi} < \mu_{k_i}^*(A_i), z' >< \infty$. Let $\Pi_{i=1}^{n} C_i = \cap_{k=1}^{\infty} \Pi_{i=1}^{n} A_{k_i}$ such that $\cap_{k=1}^{\infty} A_{k_i} = A_i \Rightarrow \Pi_{i=1}^{n} A_i \downarrow \Pi_{i=1}^{n} C_i$. ([7], p.20) and ([3], p. 92)

$LUB_k \int_n \ldots \int_1 f\delta(\Pi_{i=1}^{n})_{\Phi} < \mu_{k_i}^*(A_i), z' >\downarrow LUB_k \int_n \ldots \int_1 f\delta(\Pi_{i=1}^{n})_{\Phi} < \mu_{k_i}^*(C_i), z' >$ for each *k*,

$$LUB_k \int_n \ldots \int_1 f\delta(\Pi_{i=1}^{n})_{\Phi} < \mu_{k_i}^*(C_i), z' >= \int_n \ldots \int_1 f\delta(\Pi_{i=1}^{n})_{\Phi} < \mu_i^*(E_i), z' >$$

Proposition 5. Every measurable set $\Pi_{i=1}^{n} E_i \in \Pi_{i=1}^{n} R_i$ with σ -finite projective tensor product vector measure has a measurable cover i.e. there exists a set $N(f) = \Pi_{i=1}^{n} C_i$ in $\Pi_{i=1}^{n} G(R_i)$ such that $\int_n \ldots \int_1 f$ $\delta(\Pi_{i=1}^{n})_{\Phi} < \mu_i^*(E_i), z' >= LUB_k \int_n \ldots \int_1 f\delta(\Pi_{i=1}^{n})_{\Phi} < \mu_{k_i}^*(C_i), z' >$

Proof. By σ-finiteness of the projective tensor product vector measure, there exists a sequence $(A_{k_i})_{k=1}^{\infty}$ in $\Pi_{i=1}^{n} R_i$ such that $\int_n ....\int_1 f\delta(\Pi_{i=1}^{n})_{\Phi} < \mu_i^*(\cup_{k=1}^{\infty} A_{k_i}), z' >< \infty$ for all k and

$$\int_n ....\int_1 f\delta(\Pi_{i=1}^{n})_{\Phi} < \mu_i^*(E_i), z' > \leq \int_n ....\int_1 f\delta(\Pi_{i=1}^{n})_{\Phi} < \mu_i^*(\cup_{k=1}^{\infty} A_{k_i}), z' > .$$

Suppose

$$\int_n ....\int_1 f\delta(\Pi_{i=1}^{n})_{\Phi} < \mu_i^*(\cup_{k=1}^{\infty} A_{k_i}), z' >= \int_n ....\int_1 f\delta(\Pi_{i=1}^{n})_{\Phi} < \mu_i^*(E_i \cap (\cup_{k=1}^{\infty} A_{k_i}), z' >$$
$$\Rightarrow \int_n ....\int_1 f\delta(\Pi_{i=1}^{n})_{\Phi} < \mu_i^*(E_i), z' >= \int_n ....\int_1 f\delta(\Pi_{i=1}^{n})_{\Phi} < \mu_i^*(\cup_{k=1}^{\infty} A_{k_i}), z' > .$$

From Proposition 4, there exists a set $\Pi_{i=1}^{n} H_{k_i}$ in $\Pi_{i=1}^{n} G(R_i)$ such that $\Pi_{i=1}^{n} A_{k_i} \Theta \Pi_{i=1}^{n} H_{k_i}$. Define $\int_n ....\int_1 f\delta(\Pi_{i=1}^{n})_{\Phi} < \mu_i^*(C_i), z' >= \Sigma_{k=1}^{\infty} \int_n ....\int_1 f\delta(\Pi_{i=1}^{n})_{\Phi} < \mu_i^*(H_{k_i}), z' >$. From the results above, we obtain $\Pi_{i=1}^{n} E_i \Theta \Pi_{i=1}^{n} C_i$. Since $E_i \subset C_i$ for $1 \leq i \leq n$ hold (see [2]). Now,

$$\int_n ...\int_1 f\delta(\Pi_{i=1}^{n})_{\Phi} < \mu_i^*(E_i), z' >= LUB_k \int_n ...\int_1 f\delta(\Pi_{i=1}^{n})_{\Phi} < \mu_{k_i}^*(C_i), z' >$$

Theorem 1. If $\Pi_{i=1}^{n} E_i = N(f)$ is a $\Pi_{i=1}^{n} R_i$ - measurable set of σ-finite projective tensor product vector measure, there exists a set $\Pi_{i=1}^{n} C_i$ in $\Pi_{i=1}^{n} G(R_i)$ such that $\Pi_{i=1}^{n} E_i \Theta \Pi_{i=1}^{n} C_i$ and

$$\int_n ...\int_1 f\delta(\Pi_{k=1}^{n})_{\Phi} < \mu_i^*(C_i - E_i), z' >= 0$$

Proof. By σ-finiteness of the projective tensor product of vector measure duality, there exists a sequence $(A_{k_i})_{k=1}^{\infty}$ in $\Pi_{i=1}^{n} R_i$ such that

$$\int_n ...\int_1 f\delta(\Pi_{i=1}^{n})_{\Phi} < \mu_i^*(E_i), z' >= \int_n ...\int_1 f\delta(\Pi_{i=1}^{n})_{\Phi} < \mu_i^*(\cup_{k=1}^{\infty} A_{k_i}), z' >$$

and $< T_{(\Pi_{i=1}^{n})_{\Phi} \mu_i^*(A_{k_i})}(f), Z' >< \infty$ for all k. For each k, choose a set $\Pi_{i=1}^{n} H_i$ in $\Pi_{i=1}^{n} G(R_i)$ such that $\int_n ...\int_1 f\delta(\Pi_{i=1}^{n})_{\Phi} < \mu_i^*(A_{k_i}), z' > \leq \int_n ...\int_1 f\delta(\Pi_{i=1}^{n})_{\Phi} < \mu_i^*(H_i), z' >$. Since $N(f) = \Pi_{i=1}^{n} E_i$, it follows that $\chi_{\Pi_{i=1}^{n} H_i} f \leq f$ for $f > 0$ Define

$$\int_n ....\int_1 f\delta(\Pi_{i=1}^{n})_{\Phi} < \mu_i^*(A_{k_i}), z' >= \int_n ...\int_1 f\delta(\Pi_{i=1}^{n})_{\Phi} < \mu_i^*(E_i \cap H_i), z' > .$$

Then,

$$\int_n \cdots \int_1 f\delta(\Pi_{i=1}^n)_\Phi < \mu_i^*(E_i), z' >= \int_n \cdots \int_1 f\delta(\Pi_{i=1}^n)_\Phi < \mu_i^*(\cup_{k=1}^\infty A_{k_i}), z' >$$
$$= \Sigma_{k=1}^\infty \int_n \cdots \int_1 f\delta(\Pi_{i=1}^n)_\Phi < \mu_i^*(A_{k_i}), z' > .$$

Since $\Pi_{i=1}^n E_i$ is $\Pi_{i=1}^n R_i$-measurable, the sequence of sets $(A_{k_i})_{k=1}^\infty$ are $\Pi_{i=1}^n R_i$-measurable. Therefore,

$$\int_n \cdots\cdots \int_1 f\delta(\Pi_{i=1}^n)_\Phi < \mu_i^*(A_{k_i}), z' >< \infty.$$

By proposition 4, there exists a set $\Pi_{i=1}^n H_{k_i}$ in $\Pi_{i=1}^n G(R_i)$ such that $\Pi_{i=1}^n A_{k_i} \Theta \Pi_{i=1}^n H_{k_i}$. Define

$$\int_n \cdots\cdots \int_1 f\delta(\Pi_{i=1}^n)_\Phi < \mu_i^*(C_i), z' >= \Sigma_{k=1}^\infty \int_n \cdots\cdots \int_1 f\delta(\Pi_{i=1}^n)_\Phi < \mu_i^*(H_{k_i}), z' > .$$

From the results above, we have $\Pi_{i=1}^n E_i \Theta \Pi_{i=1}^n C_i$.

So, $\int_n \cdots\cdots \int_1 f\delta(\Pi_{i=1}^n)_\Phi < \mu_i^*(C_i - E_i), z' >= \int_n \cdots\cdots \int_1 f\delta(\Pi_{i=1}^n)_\Phi < \mu_i^*(\cup_{k=1}^\infty (H_{k_i} - A_{k_i}), z')$. Since, $\Pi_{i=1}^n A_{k_i} \Theta \Pi_{i=1}^n H_{k_i}$, it follows that $\int_n \cdots \int_1 f\delta(\Pi_{i=1}^n)_\Phi < \mu_i^*(C_i - E_i), z' >= 0.$

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